

# Snark Designs

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## Abstract

The main aim of this paper is to solve the design spectrum problem for Tietze's graph, the two 18-vertex Blanuša snarks, the six snarks on 20 vertices (including the flower snark J5), the twenty snarks on 22 vertices (including the two Loupekine snarks) and Goldberg's snark #3. Together with the Petersen graph (for which the spectrum has already been computed) this list includes all non-trivial snarks of up to 22 vertices. We also give partial results for a selection of larger graphs: the two Celmins-Swart snarks, the 26- and 34-vertex Blanuša snarks, the flower snark J7, the double star snark, Zamfirescu's graph, Goldberg's snark #5, the Szekeres snark and the Watkins snark.

## 1 Introduction

Throughout this paper all graphs are simple. Let  $G$  be a graph. If the edge set of a graph  $K$  can be partitioned into edge sets of graphs each isomorphic to  $G$ , we say that there exists a *decomposition* of  $K$  into  $G$ . In the case where  $K$  is the complete graph  $K_n$  we refer to the decomposition as a  $G$  *design* of order  $n$ . The *spectrum* of  $G$  is the set of positive integers  $n$  for which there exists a  $G$  design of order  $n$ . For completeness we remark here that the empty set is a  $G$  design of order 1, and that this trivial case will be omitted from discussion henceforth.

A complete solution of the spectrum problem often seems to be very difficult. However it has been achieved in many cases, especially amongst the smaller graphs. We refer the reader to the survey article of Adams, Bryant and Buchanan, [4] and, for more up to date results, the Web site maintained by Bryant and McCourt, [6]. In particular, as a result of recent

work on the graphs of the Platonic solids, [5], [14], the spectrum of the dodecahedron is now known and for the icosahedron the spectrum has been determined up to a small number of unresolved cases. The Archimedean solids have also received recent attention, [12], [13], and we now have known spectra for five of the graphs and partial results for a further seven. As far as the author is aware, the current state of knowledge is as follows.

1. Tetrahedron designs of order  $n$  exist if and only if  $n \equiv 1$  or  $4 \pmod{12}$ , [20].
2. Octahedron designs of order  $n$  exist if and only if  $n \equiv 1$  or  $9 \pmod{24}$ ,  $n \neq 9$ , [19], [1].
3. Cube designs of order  $n$  exist if and only if  $n \equiv 1$  or  $16 \pmod{24}$ , [22], [21], [8].
4. Dodecahedron designs of order  $n$  exist if and only if  $n \equiv 1, 16, 25$  or  $40 \pmod{60}$  and  $n \neq 16$ , [2], [4], [5].
5. Icosahedron designs of order  $n$  exist if  $n \equiv 1, 16, 21$  or  $36 \pmod{60}$  except possibly  $n = 21, 141, 156, 201, 261, 276$ ; [2], [4], [14].
6. Cuboctahedron designs of order  $n$  exist if and only if  $n \equiv 1$  or  $33 \pmod{48}$ , [17].
7. Rhombicuboctahedron designs of order  $n$  exist if and only if  $n \equiv 1$  or  $33 \pmod{96}$ , [11].
8. Truncated tetrahedron design of order  $n$  exist if and only if  $n \equiv 1$  or  $28 \pmod{36}$ , [12].
9. Truncated octahedron designs of order  $n$  exist if and only if  $n \equiv 1$  or  $64 \pmod{72}$ , [12].
10. Truncated cube designs of order  $n$  exist if and only if  $n \equiv 1$  or  $64 \pmod{72}$ , [12].
11. Truncated cuboctahedron, icosidodecahedron, rhombicosidodecahedron, truncated icosahedron, truncated dodecahedron, truncated icosidodecahedron and snub cube designs of order  $n$  exist if  $n \equiv 1 \pmod{2e}$ , where  $e$  is the number of edges in the Archimedean graph [13].
12. There exists an icosidodecahedron design of order 81 and a truncated dodecahedron design of order 145, [13].

Table 1: Design existence conditions for 3-regular graphs

$v$	typical graphs	conditions
10	Petersen graph	$n \equiv 1 \text{ or } 10 \pmod{15}$
12	Tietze graph	$n \equiv 1 \text{ or } 28 \pmod{36}$
18	Blanuša snarks	$n \equiv 1 \pmod{27}$
20	Flower snark J5	$n \equiv 1, 16, 25, 40 \pmod{60}, n \neq 16$
22	Loupevine snarks	$n \equiv 1 \text{ or } 22 \pmod{33}$
24	Goldberg's snark #3	$n \equiv 1 \text{ or } 64 \pmod{72}$
26	Celmins–Swart snarks Blanuša snarks	$n \equiv 1 \text{ or } 13 \pmod{39}, n \neq 13$
28	Flower snark J7	$n \equiv 1, 28, 49, 64 \pmod{84}$
30	Double star snark	$n \equiv 1 \text{ or } 10 \pmod{45}, n \neq 10$
34	Blanuša snarks	$n \equiv 1 \text{ or } 34 \pmod{51}$
36	Zamfirescu's graph	$n \equiv 1 \text{ or } 28 \pmod{108}, n \neq 28$
40	Goldberg's snark #5	$n \equiv 1, 16, 25, 40 \pmod{120}, n \neq 16, 25$
50	Szekeres & Watkins snarks	$n \equiv 1 \text{ or } 25 \pmod{75}, n \neq 25$

A *snark* is a connected, bridgeless 3-regular graph with chromatic index 4. However, a snark is usually regarded as trivial (or reducible) if it has girth less than 5 or if it has three edges the deletion of which results in a disconnected graph each of whose components is non-trivial (as a graph). The term is due to Martin Gardner who, in a popular account of the subject, [15], likened the elusive nature of these graphs to that of the quarry in Lewis Carroll's poem, [9]. It appears that the only non-trivial snark where the spectrum has been found is the smallest, namely the Petersen graph, [3]. The main purpose of this paper is to determine the spectrum for each of the remaining non-trivial snarks with up to 22 vertices: the two Blanuša snarks on 18 vertices, the six snarks on 20 vertices and the twenty snarks on 22 vertices (Theorems 1.2, 1.3 and 1.4). There are no non-trivial snarks on 12, 14 or 16 vertices. In addition, we compute the spectra of two trivial snarks, the 12-vertex Tietze's graph (Theorem 1.1) and the 24-vertex Goldberg's snark #3 (Theorem 1.5), and we report partial results for some of the larger non-trivial snarks (Theorem 1.6).

Let  $v$  denote the number of vertices of  $G$ . It is clear that for a 3-regular graph  $G$ , a  $G$  design of order  $n$  can exist only if (i)  $n = 1$  or  $n \geq v$ , (ii)  $n(n-1) \equiv 0 \pmod{3v}$  and (iii)  $n \equiv 1 \pmod{3}$ . These conditions are determined by elementary counting and given explicitly in Table 1 for some values of  $v$ . We now state our results.

**Theorem 1.1** *Designs of order  $n$  exist for the Tietze graph if and only if*

$n \equiv 1$  or  $28 \pmod{36}$ .

**Theorem 1.2** *Designs of order  $n$  exist for each of the two 18-vertex Blanuša snarks if and only if  $n \equiv 1 \pmod{27}$ .*

**Theorem 1.3** *Designs of order  $n$  exist for each of the six snarks on 20 vertices if and only if  $n \equiv 1, 16, 25$  or  $40 \pmod{60}$  and  $n \neq 16$ .*

**Theorem 1.4** *Designs of order  $n$  exist for each of the twenty snarks on 22 vertices if and only if  $n \equiv 1$  or  $22 \pmod{33}$ .*

**Theorem 1.5** *Designs of order  $n$  exist for Goldberg's snark #3 if and only if  $n \equiv 1$  or  $64 \pmod{72}$ .*

Our last theorem is a collection of partial results, with half of the possible residue classes being resolved in each case.

**Theorem 1.6** (i) *Designs of order  $n$  exist for the two Celmins-Swart snarks and the two 26-vertex Blanuša snarks if  $n \equiv 1 \pmod{39}$ .*

(ii) *Designs of order  $n$  exist for the flower snark  $J7$  if  $n \equiv 1$  or  $28 \pmod{84}$ .*

(iii) *Designs of order  $n$  exist for the double star snark if  $n \equiv 1 \pmod{45}$ .*

(iv) *Designs of order  $n$  exist for the two 34-vertex Blanuša snarks if  $n \equiv 1 \pmod{51}$ .*

(v) *Designs of order  $n$  exist for Zamfirescu's graph if  $n \equiv 1 \pmod{108}$ .*

(vi) *Designs of order  $n$  exist for Goldberg's snark #5 if  $n \equiv 1$  or  $40 \pmod{120}$ .*

(vii) *Designs of order  $n$  exist for both the Szekeres snark and the Watkins snark if  $n \equiv 1 \pmod{75}$ .*

Our method of proof uses Wilson's fundamental construction. For this we need the concept of a *group divisible design* (GDD). Recall that a  $k$ -GDD of type  $g^t$  is an ordered triple  $(V, \mathcal{G}, \mathcal{B})$  where  $V$  is a base set of cardinality  $v = tg$ ,  $\mathcal{G}$  is a partition of  $V$  into  $t$  subsets of cardinality  $g$  called *groups* and  $\mathcal{B}$  is a collection of subsets of cardinality  $k$  called *blocks* which collectively have the property that each pair of elements from different groups occurs in precisely one block but no pair of elements from the same group occurs at all. We will also need  $k$ -GDDs of type  $g^t h^1$  where  $h \neq g$ . These are defined analogously, with the base set  $V$  having cardinality  $tg + h$  and partitioned into  $t$  subsets of cardinality  $g$  and one subset of cardinality  $h$ . In the propositions of this section and elsewhere we give primary references for the existence of the 3- and 4-GDDs that we use in our constructions. The same information is conveniently presented in Ge's article in the *Handbook of Combinatorial Designs*, [16]. As is well known, a  $k$ -GDD of type  $g^k$  exists

whenever there exist  $k - 2$  mutually orthogonal Latin squares (MOLS) of side  $g$ .

The remaining sections deal with each of the snarks stated in the theorems. We give actual designs for a few chosen numbers as well as decompositions of certain multipartite graphs. These are used in combination with group divisible designs to construct the complete graphs required to prove the theorems. The decompositions were created and checked by a computer program written in the C language. They were checked again by a simple MATHEMATICA program. A further check was made by a more sophisticated MATHEMATICA program which reads the text of the paper and extracts for appropriate analysis anything that looks like a graph edge set or a graph decomposition. It is hoped that this threefold verification, perhaps reinforced by the Bellman's assertion, "What I tell you three times is true", [9], will help persuade the reader that the results presented in Sections 2–13 are correct.

For our purposes a graph  $G$  has vertices  $V(G) = \{1, 2, \dots, v\}$  for some positive integer  $v$  and, since there are no isolated vertices, it is defined by its edge set  $E(G) = \{\{i, j\} : i, j \in V(G), i \sim j\}$ . In the main body of the paper we specify the edge sets that we actually used in our computations. Pictures of some of the graphs can be found in [25], [24] and [23]. If  $G$  has  $e$  edges, the numbers of occurrences of  $G$  in a decomposition into  $G$  of the complete graph  $K_n$ , the complete  $r$ -partite graph  $K_{n^r}$  and the complete  $(r + 1)$ -partite graph  $K_{n^r m^1}$  are respectively

$$\frac{n(n-1)}{2e}, \quad \frac{n^2 r(r-1)}{2e} \quad \text{and} \quad \frac{nr(n(r-1) + 2m)}{2e}.$$

**Proposition 1.1** *Let  $G$  be a graph with  $e > 0$  edges. Suppose there exist  $G$  designs of order  $2e + 1$  and  $4e + 1$ . Suppose also there exists a decomposition into  $G$  of the complete tripartite graph  $K_{e,e,e}$ . Then there exist  $G$  designs of order  $n$  for  $n \equiv 1 \pmod{2e}$ .*

**Proof.** There exist 3-GDDs of types  $2^{3t}$  and  $2^{3t+1}$  for  $t \geq 1$ , [20] (see also [16]). In each case we perform the following construction. Replace each point of the 3-GDD by  $e$  elements; that is, inflate by a factor of  $e$ . Add a further point, which we denote by the symbol  $\infty$ . Lay a complete graph  $K_{2e+1}$  on each of the inflated groups together with  $\infty$  and replace each block of the 3-GDD by a complete tripartite graph  $K_{e,e,e}$ . This yields designs of order  $6et + 1$  for  $t \geq 1$  in the first case and  $6et + 2e + 1$  for  $t \geq 1$  in the second case.

There exists a 3-GDD of type  $6^t 4^1$  for  $t \geq 3$ , [10] (see also [16]). Inflate by a factor of  $e$ , add  $\infty$ , lay a complete graph  $K_{6e+1}$  (from the previous construction) or  $K_{4e+1}$  on each of the inflated groups together with  $\infty$  and

replace each block of the 3-GDD by a complete tripartite graph  $K_{e,e,e}$  to give designs of order  $6et + 4e + 1$  for  $t \geq 3$ .

Finally, there exist 3-GDDs of types  $2^3 4^1$  and  $4^4$  (see [16]). Inflate by a factor of  $e$ , add  $\infty$ , lay a complete graph  $K_{2e+1}$  or  $K_{4e+1}$  on each of the inflated groups together with  $\infty$  and replace each block of the 3-GDD by a complete tripartite graph  $K_{e,e,e}$  to give designs of orders  $10e + 1$  and  $16e + 1$ , the two orders not covered by the previous constructions.  $\square$

**Proposition 1.2** *Let  $G$  be a graph with  $e > 0$  edges,  $3|e$ . Suppose there exist  $G$  designs of order  $2e + 1$  and  $4e + 1$ . Suppose also there exists a decomposition into  $G$  of the complete tripartite graph  $K_{e/3,e/3,e/3}$ . Then there exist  $G$  designs of order  $n$  for  $n \equiv 1 \pmod{2e}$ .*

**Proof.** There exists a 3-GDD of type  $3^3$  (Latin square of side 3). Inflate by a factor of  $e/3$ , and replace each block by a complete tripartite graph  $K_{e/3,e/3,e/3}$  to yield a complete tripartite graph  $K_{e,e,e}$ . Now use Proposition 1.1.  $\square$

**Proposition 1.3** *Let  $G$  be a graph with  $e > 0$  edges,  $3|e$ . Suppose there exist  $G$  designs of order  $e + 1$ ,  $2e + 1$ , and  $4e + 1$ . Suppose also there exists a decomposition into  $G$  of the complete tripartite graph  $K_{e/3,e/3,e/3}$ . Then there exist  $G$  designs of order  $n$  for  $n \equiv 1 \pmod{e}$ .*

**Proof.** Take a 3-GDD of type  $6^t$ ,  $t \geq 3$ , [20] (see also [16]). Inflate by a factor of  $e/3$ , add a further point,  $\infty$ , lay a complete graph  $K_{2e+1}$  on each of the inflated groups together with  $\infty$  and replace each block of the 3-GDD by a complete tripartite graph  $K_{e/3,e/3,e/3}$ . This gives a  $G$  design of order  $2et + 1$  for  $t \geq 3$ .

There exists a 3-GDD of type  $3^{2t+1}$  for  $t \geq 1$ , [20] (see also [16]). Inflate by a factor of  $e/3$ , add  $\infty$ , lay a complete graph  $K_{e+1}$  on each of the inflated groups together with  $\infty$  and replace each block of the 3-GDD by a complete tripartite graph  $K_{e/3,e/3,e/3}$ . This gives a  $G$  design of order  $2et + e + 1$  for  $t \geq 1$ .  $\square$

**Proposition 1.4** *Let  $G$  be a graph with  $e > 0$  edges,  $3|e$ . Suppose there exist  $G$  designs of order  $e + 1$  and  $2e + 1$ . Suppose also there exist decompositions into  $G$  of the complete tripartite graph  $K_{e,e,e}$  and the complete 4-partite graph  $K_{e/3,e/3,e/3,e/3}$ . Then there exist  $G$  designs of order  $n$  for  $n \equiv 1 \pmod{e}$ .*

**Proof.** There exists a 4-GDD of type  $6^t$  for  $t \geq 5$ , [7] (see also [16]). Inflate by a factor of  $e/3$ , add  $\infty$ , lay a complete graph  $K_{2e+1}$  on each of the inflated groups together with  $\infty$  and replace each block by a complete 4-partite graph  $K_{e/3, e/3, e/3, e/3}$ . This gives a  $G$  design of order  $2et + 1$  for  $t \geq 5$ .

There exists a 4-GDD of type  $6^t 3^1$  for  $t \geq 4$ , [18] (see also [16]). Inflate by a factor of  $e/3$ , add  $\infty$ , lay a complete graph  $K_{e+1}$  or  $K_{2e+1}$  on each of the inflated groups together with  $\infty$  and replace each block by a complete 4-partite graph  $K_{e/3, e/3, e/3, e/3}$ . This gives a  $G$  design of order  $2et + e + 1$  for  $t \geq 4$ .

Using 4-GDDs of types  $3^4$ ,  $3^5$ ,  $3^5 6^1$  and  $3^8$  (see [16]) we proceed as above to yield designs of order  $4e + 1$ ,  $5e + 1$ ,  $7e + 1$  and  $8e + 1$  respectively.

To deal with the remaining cases we use a 3-GDD of type  $1^3$  or  $2^3$  (Latin square of side 2). Inflate by a factor of  $e$ , add  $\infty$ , lay a complete graph  $K_{e+1}$  or  $K_{2e+1}$  on each of the inflated groups together with  $\infty$  and replace each block by a complete tripartite graph  $K_{e, e, e}$  to produce a design of order  $3e + 1$  or  $6e + 1$ .  $\square$

**Proposition 1.5** *Let  $G$  be a 3-regular graph on 12 vertices. Suppose there exist  $G$  designs of order 28, 37, 64, 73 and 100. Suppose also that there exist decompositions into  $G$  of the complete multipartite graphs  $K_{6,6,6}$  and  $K_{3,3,3,3}$ . Then there exists a  $G$  design of order  $n$  if and only if  $n \equiv 1$  or  $28 \pmod{36}$ .*

**Proof.** The truncated tetrahedron is a 3-regular graph on 12 vertices, which is known to have the spectrum stated in the proposition, [12]. Since the constructions in [12] for the truncated tetrahedron use precisely the decompositions stated in this proposition, any graph  $G$  satisfying all the conditions of the proposition has the same spectrum.  $\square$

**Proposition 1.6** *Let  $G$  be a 3-regular graph on 20 vertices. Suppose there exist  $G$  designs of order 25, 40, 61, 76, 85, 121, 160, 181 and 220. Suppose also that there exist decompositions into  $G$  of the complete multipartite graphs  $K_{5,5,5,5}$ ,  $K_{10,10,10,10}$ ,  $K_{6,6,6,6,6,6}$ ,  $K_{60,60,75}$ ,  $K_{60,60,60,75}$ ,  $K_{15,15,15,21}$ ,  $K_{24,24,24,24,39}$  and  $K_{24,24,24,24,24,24,60}$ . Then there exists a  $G$  design of order  $n$  if and only if  $n \equiv 1, 16, 25$  or  $40 \pmod{60}$  and  $n \neq 16$ .*

**Proof.** The dodecahedron is a 3-regular graph on 20 vertices, which is known to have the spectrum stated in the proposition, [5]. We therefore follow the proof given in [5], except that the sporadic value 340 of [5, Section 4] is handled in a different way. For this purpose an additional

decomposition into  $G$ , namely that of  $K_{10,10,10,10}$ , is required. The other decompositions stated in the proposition correspond to those used by the constructions described in [5].

There exists a 4-GDD of type  $4^6 10^1$ , [18] (see also [16]). Inflate by a factor of 10, lay a complete graph  $K_{40}$  or  $K_{100}$  (for the construction of the latter, see [5, Proposition 3.3]) on each of the inflated groups and replace each block of the 4-GDD by a complete 4-partite graph  $K_{10,10,10,10}$ . This construction yields a  $G$  design of order 340.  $\square$

**Proposition 1.7** *Let  $G$  be a 3-regular graph on 22 vertices. Suppose there exist  $G$  designs of order 22, 34, 55, 67 and 88. Suppose also that there exist decompositions into  $G$  of the complete multipartite graphs  $K_{33,33,33}$ ,  $K_{11,11,11,11}$ ,  $K_{22,22,22,55}$  and  $K_{22,22,22,22,22,55}$ . Then there exists a  $G$  design of order  $n$  if and only if  $n \equiv 1$  or  $22 \pmod{33}$ .*

**Proof.** The ‘only if’ part follows from the standard divisibility conditions—see Table 1. Decompositions of  $K_{34}$ ,  $K_{67}$ ,  $K_{33,33,33}$  and  $K_{11,11,11,11}$  together with Proposition 1.4 yield designs of order  $1 \pmod{33}$ .

There exist 4-GDDs of type  $2^{3t+1}$  for  $t \geq 2$  and type  $2^{3t}5^1$  for  $t \geq 3$ , [7], [18] (see also [16]). In each case, inflate by a factor of 11, lay a complete graph  $K_{22}$  or  $K_{55}$  on each of the inflated groups and replace each block by a complete 4-partite graph  $K_{11,11,11,11}$ . Since decompositions exist for  $K_{22}$ ,  $K_{55}$  and  $K_{11,11,11,11}$ , this construction yields designs of order  $22 \pmod{33}$  except for 88, 121 and 187.

For 121 and 187 we use the decompositions of  $K_{22}$ ,  $K_{55}$ ,  $K_{22,22,22,55}$  and  $K_{22,22,22,22,22,55}$  to construct decompositions of the complete graphs  $K_{121}$  and  $K_{187}$ .  $\square$

**Proposition 1.8** *Let  $G$  be a 3-regular graph on 24 vertices. Suppose there exist  $G$  designs of order 64, 73, 136 and 145. Suppose also that there exist decompositions into  $G$  of the complete multipartite graphs  $K_{12,12,12}$ ,  $K_{24,24,15}$ ,  $K_{72,72,63}$ ,  $K_{24,24,24,24}$  and  $K_{24,24,24,21}$ . Then there exists a  $G$  design of order  $n$  if and only if  $n \equiv 1$  or  $64 \pmod{72}$ .*

**Proof.** The ‘only if’ part follows from the standard divisibility conditions—see Table 1. Decompositions of  $K_{73}$ ,  $K_{145}$  and  $K_{12,12,12}$  together with Proposition 1.2 yield designs of order  $1 \pmod{72}$ .

For the other residue class, we first construct a decomposition into  $G$  of the complete tripartite graph  $K_{24,24,24}$  using the decomposition of  $K_{12,12,12}$  and a 3-GDD of type  $2^3$  (Latin square of side 2), as in the proof of Proposition 1.2.



There exists a 4-GDD of type  $6^t 3^1$  for  $t \geq 4$ , [18] (see also [16]). Inflate the 3-element group by a factor of 21, inflate all other groups by a factor of 24 and add a new point,  $\infty$ . Lay a complete graph  $K_{64}$  over the inflated 3-element group together with  $\infty$  and lay a complete graph  $K_{145}$  over each of the other inflated groups together with  $\infty$ . Replace each block of the 4-GDD by one of the complete 4-partite graphs  $K_{24,24,24,24}$  or  $K_{24,24,24,21}$ , as appropriate. Since decompositions exist for  $K_{64}$ ,  $K_{145}$ ,  $K_{24,24,24,24}$  and  $K_{24,24,24,21}$ , this construction yields designs of order  $64 \pmod{144}$  except for 208, 352 and 496.

There exists a 3-GDD of type  $3^{2t} 9^1$  for  $t \geq 2$ , [10] (see also [16]). Inflate the 9-element group by a factor of 15, inflate all other groups by a factor of 24, add  $\infty$ , lay a complete graph  $K_{136}$  over the inflated 9-element group together with  $\infty$  and lay a complete graph  $K_{73}$  over each of the other groups together with  $\infty$ . Replace each block of the 3-GDD by one of the complete tripartite graphs  $K_{24,24,24}$  or  $K_{24,24,15}$ , as appropriate. Since decompositions exist for  $K_{73}$ ,  $K_{136}$ ,  $K_{24,24,24}$  and  $K_{24,24,15}$ , this construction yields designs of order  $136 \pmod{144}$  except for 280.

It only remains to deal with the four cases not covered by the previous constructions. For 208, we construct a complete graph  $K_{208}$  by augmenting the complete tripartite graph  $K_{72,72,63}$  with a common point,  $\infty$ , and laying complete graphs  $K_{73}$  and  $K_{64}$  over the augmented partitions.

There exist 4-GDDs of types  $3^4$ ,  $3^5$  and  $3^5 6^1$  (see [16]). In each case, inflate one of the 3-element groups by a factor of 21, inflate all other groups by a factor of 24, add  $\infty$ , lay a complete graph  $K_{64}$  over the 21-inflated 3-element group together with  $\infty$ , lay a complete graph  $K_{73}$  over each of the other inflated 3-element groups together with  $\infty$  and, if present, lay a complete graph  $K_{145}$  over the inflated 6-element group together with  $\infty$ . Replace each block of the 3-GDD by  $K_{24,24,24,24}$  or  $K_{24,24,24,21}$ , as appropriate. These constructions give  $G$  designs of orders 280, 352 and 496 respectively.  $\square$

## 2 The Tietze graph

The Tietze graph is represented by the ordered 12-tuple of its vertices  $(1, 2, \dots, 12)_T$ . The edge set that we use, as supplied with the MATHEMATICA system, [24], is  $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{8, 9\}, \{1, 9\}, \{1, 10\}, \{4, 11\}, \{7, 12\}, \{3, 8\}, \{2, 6\}, \{5, 9\}, \{10, 11\}, \{11, 12\}, \{10, 12\}\}$ .

**Lemma 2.1** *There exist Tietze graph designs of order 28, 37, 64, 73 and 100.*

**Proof.** Let the vertex set of  $K_{28}$  be  $Z_{28}$ . The decomposition consists of the graphs

$$\begin{aligned} &(23, 26, 13, 6, 0, 1, 2, 3, 4, 7, 11, 18)_T, \\ &(0, 2, 4, 1, 3, 8, 11, 13, 17, 10, 6, 19)_T, \\ &(0, 5, 11, 1, 13, 20, 12, 24, 7, 18, 9, 26)_T \end{aligned}$$

under the action of the mapping  $x \mapsto x + 4 \pmod{28}$ .

Let the vertex set of  $K_{37}$  be  $Z_{37}$ . The decomposition consists of the graph

$$(0, 1, 3, 6, 2, 7, 14, 23, 15, 10, 22, 33)_T$$

under the action of the mapping  $x \mapsto x + 1 \pmod{37}$ .

Let the vertex set of  $K_{64}$  be  $Z_{63} \cup \{\infty\}$ . The decomposition consists of the graphs

$$\begin{aligned} &(\infty, 6, 30, 11, 56, 16, 62, 39, 41, 19, 22, 24)_T, \\ &(30, 26, 7, 33, 58, 32, 29, 3, 15, 14, 55, 42)_T, \\ &(20, 52, 50, 25, 58, 13, 23, 37, 29, 51, 5, 56)_T, \\ &(44, 3, 43, 49, 7, 2, 36, 54, 24, 17, 56, 52)_T, \\ &(10, 61, 52, 25, 39, 33, 12, 4, 3, 54, 57, 2)_T, \\ &(1, 2, 9, 43, 44, 51, 22, 23, 30, 17, 59, 38)_T \end{aligned}$$

under the action of the mapping  $\infty \mapsto \infty$ ,  $x \mapsto x + 3 \pmod{63}$  for the first five, and  $x \mapsto x + 9 \pmod{63}$  for the last one.

Let the vertex set of  $K_{73}$  be  $Z_{73}$ . The decomposition consists of the graphs

$$\begin{aligned} &(0, 1, 3, 6, 2, 7, 14, 22, 13, 10, 24, 36)_T, \\ &(0, 15, 31, 1, 18, 51, 26, 70, 38, 23, 47, 68)_T \end{aligned}$$

under the action of the mapping  $x \mapsto x + 1 \pmod{73}$ .

Let the vertex set of  $K_{100}$  be  $Z_{100}$ . The decomposition consists of the graphs

$$\begin{aligned} &(10, 43, 83, 87, 79, 15, 80, 92, 63, 46, 73, 35)_T, \\ &(81, 7, 5, 21, 12, 92, 11, 87, 49, 84, 71, 25)_T, \\ &(71, 14, 84, 79, 72, 82, 19, 22, 28, 61, 37, 41)_T, \\ &(34, 38, 24, 32, 74, 50, 6, 77, 47, 59, 91, 80)_T, \\ &(6, 91, 47, 96, 21, 86, 12, 2, 85, 54, 17, 69)_T, \\ &(87, 20, 33, 64, 16, 44, 98, 32, 65, 60, 22, 91)_T, \\ &(89, 30, 50, 78, 81, 39, 18, 41, 42, 12, 94, 16)_T, \\ &(36, 34, 74, 68, 4, 21, 29, 69, 97, 35, 52, 86)_T, \\ &(58, 40, 79, 38, 15, 63, 2, 13, 25, 50, 69, 80)_T, \\ &(5, 20, 65, 10, 60, 6, 87, 58, 23, 59, 85, 12)_T, \\ &(64, 45, 43, 55, 49, 79, 59, 42, 93, 96, 61, 56)_T \end{aligned}$$

under the action of the mapping  $x \mapsto x + 4 \pmod{100}$ . □

**Lemma 2.2** *There exist decompositions of the complete multipartite graphs  $K_{6,6,6}$  and  $K_{3,3,3,3}$  into the Tietze graph.*

**Proof.** Let the vertex set of  $K_{6,6,6}$  be  $Z_{18}$  partitioned according to residue classes modulo 3. The decomposition consists of the graph

$$(0, 1, 2, 3, 7, 11, 4, 9, 5, 8, 13, 6)_T$$

under the action of the mapping  $x \mapsto x + 3 \pmod{18}$ .

Let the vertex set of  $K_{3,3,3,3}$  be  $Z_{12}$  partitioned according to residue classes modulo 4. The decomposition consists of the graph

$$(0, 1, 2, 3, 4, 6, 9, 7, 10, 5, 8, 11)_T$$

under the action of the mapping  $x \mapsto x + 4 \pmod{12}$ .  $\square$

Theorem 1.1 follows from Lemmas 2.1, 2.2 and Proposition 1.5.

### 3 The two 18-vertex Blanuša snarks

The 18-vertex Blanuša snarks are represented by the ordered 18-tuples  $(1, 2, \dots, 18)_{B11}$  for the  $(1, 1)$ -Blanuša snark and  $(1, 2, \dots, 18)_{B12}$  for the  $(1, 2)$ -Blanuša snark. The edge sets, as supplied with the MATHEMATICA system, [24], are respectively

B11:  $\{\{2, 4\}, \{3, 4\}, \{1, 2\}, \{1, 5\}, \{3, 5\}, \{6, 8\}, \{16, 18\}, \{16, 17\}, \{6, 7\}, \{2, 7\}, \{1, 8\}, \{4, 9\}, \{5, 10\}, \{11, 15\}, \{14, 15\}, \{12, 14\}, \{12, 13\}, \{11, 13\}, \{14, 17\}, \{13, 16\}, \{15, 18\}, \{8, 9\}, \{7, 10\}, \{9, 11\}, \{6, 12\}, \{3, 18\}, \{10, 17\}\}$  and

B12:  $\{\{2, 4\}, \{3, 4\}, \{1, 2\}, \{1, 5\}, \{3, 5\}, \{3, 6\}, \{6, 8\}, \{8, 18\}, \{16, 18\}, \{16, 17\}, \{7, 17\}, \{6, 7\}, \{2, 7\}, \{1, 8\}, \{4, 9\}, \{9, 12\}, \{5, 10\}, \{10, 11\}, \{11, 15\}, \{14, 15\}, \{12, 14\}, \{12, 13\}, \{11, 13\}, \{14, 17\}, \{13, 16\}, \{15, 18\}, \{9, 10\}\}$ .

**Lemma 3.1** *There exist designs of order 28, 55 and 109 for each of the two 18-vertex Blanuša snarks.*

**Proof.** Let the vertex set of  $K_{28}$  be  $Z_{28}$ . The decompositions consist of

$$(0, 1, 2, 3, 4, 5, 6, 8, 7, 9, 10, 11, 16, 13, 17, 18, 23, 26)_{B11},$$

$$(0, 3, 1, 9, 12, 2, 14, 11, 27, 25, 19, 16, 26, 23, 4, 20, 10, 13)_{B11}$$

and

$$(0, 1, 2, 3, 4, 5, 6, 8, 7, 9, 13, 10, 19, 14, 20, 11, 21, 22)_{B12},$$

$$(0, 3, 1, 9, 10, 11, 20, 16, 24, 23, 7, 2, 14, 12, 21, 22, 27, 5)_{B12}$$

under the action of the mapping  $x \mapsto x + 4 \pmod{28}$ .

Let the vertex set of  $K_{55}$  be  $Z_{55}$ . The decompositions consist of

$$(0, 1, 2, 4, 6, 3, 8, 11, 20, 18, 5, 21, 38, 53, 25, 13, 39, 44)_{B11}$$

and

$$(0, 1, 2, 4, 6, 7, 14, 15, 13, 23, 3, 24, 36, 8, 22, 12, 37, 49)_{B12}$$

under the action of the mapping  $x \mapsto x + 1 \pmod{55}$ .

Let the vertex set of  $K_{109}$  be  $Z_{109}$ . The decompositions consist of

$$(87, 66, 38, 101, 5, 88, 85, 75, 3, 86, 61, 71, 0, 1, 6, 2, 8, 16)_{B11},$$

$(0, 4, 1, 19, 9, 2, 27, 26, 55, 59, 3, 81, 47, 28, 92, 7, 75, 50)_{B11}$

and

$(36, 48, 25, 43, 35, 78, 76, 2, 64, 58, 89, 93, 0, 1, 4, 7, 15, 29)_{B12},$

$(0, 4, 1, 36, 37, 10, 23, 40, 21, 63, 3, 62, 73, 8, 55, 28, 74, 99)_{B12}$

under the action of the mapping  $x \mapsto x + 1 \pmod{109}$ .  $\square$

**Lemma 3.2** *There exist decompositions of  $K_{9,9,9}$  into each of the two 18-vertex Blanuša snarks.*

**Proof.** Let the vertex set of  $K_{9,9,9}$  be  $Z_{27}$  partitioned according to residue classes modulo 3. The decompositions consist of

$(0, 1, 2, 3, 4, 5, 12, 19, 20, 23, 24, 10, 26, 14, 7, 16, 9, 21)_{B11}$

and

$(0, 1, 2, 3, 4, 6, 8, 11, 10, 18, 5, 21, 16, 14, 24, 17, 22, 7)_{B12}$

under the action of the mapping  $x \mapsto x + 3 \pmod{27}$ .  $\square$

Theorem 1.2 follows from Lemmas 3.1, 3.2 and Proposition 1.3.

## 4 The six snarks on 20 vertices

The six snarks on 20 vertices are represented by the ordered 20-tuples  $(1, 2, \dots, 20)_{S1}, (1, 2, \dots, 20)_{S2}, \dots, (1, 2, \dots, 20)_{S6}$ , with edge sets

S1:  $\{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{5, 6\}, \{6, 7\}, \{6, 8\}, \{9, 10\}, \{10, 11\}, \{10, 12\}, \{13, 14\}, \{14, 15\}, \{14, 16\}, \{17, 18\}, \{18, 20\}, \{18, 19\}, \{1, 17\}, \{13, 17\}, \{5, 9\}, \{9, 13\}, \{1, 5\}, \{3, 20\}, \{16, 19\}, \{4, 7\}, \{8, 11\}, \{12, 15\}, \{3, 8\}, \{7, 12\}, \{11, 16\}, \{15, 20\}, \{4, 19\}\},$

S2:  $\{\{1, 2\}, \{1, 3\}, \{1, 5\}, \{2, 4\}, \{2, 11\}, \{3, 10\}, \{3, 18\}, \{4, 6\}, \{4, 8\}, \{5, 7\}, \{5, 8\}, \{6, 7\}, \{6, 14\}, \{7, 9\}, \{8, 10\}, \{9, 10\}, \{9, 12\}, \{11, 12\}, \{11, 17\}, \{12, 13\}, \{13, 16\}, \{13, 19\}, \{14, 15\}, \{14, 19\}, \{15, 16\}, \{15, 17\}, \{16, 18\}, \{17, 20\}, \{18, 20\}, \{19, 20\}\},$

S3:  $\{\{1, 2\}, \{1, 4\}, \{1, 7\}, \{2, 3\}, \{2, 8\}, \{3, 10\}, \{3, 17\}, \{4, 9\}, \{4, 10\}, \{5, 6\}, \{5, 9\}, \{5, 13\}, \{6, 7\}, \{6, 11\}, \{7, 19\}, \{8, 9\}, \{8, 12\}, \{10, 12\}, \{11, 12\}, \{11, 14\}, \{13, 16\}, \{13, 18\}, \{14, 15\}, \{14, 18\}, \{15, 16\}, \{15, 17\}, \{16, 19\}, \{17, 20\}, \{18, 20\}, \{19, 20\}\},$

S4:  $\{\{1, 2\}, \{1, 3\}, \{1, 7\}, \{2, 4\}, \{2, 8\}, \{3, 9\}, \{3, 10\}, \{4, 5\}, \{4, 17\}, \{5, 6\}, \{5, 9\}, \{6, 7\}, \{6, 11\}, \{7, 19\}, \{8, 9\}, \{8, 12\}, \{10, 12\}, \{10, 13\}, \{11, 12\}, \{11, 14\}, \{13, 16\}, \{13, 18\}, \{14, 15\}, \{14, 18\}, \{15, 16\}, \{15, 17\}, \{16, 19\}, \{17, 20\}, \{18, 20\}, \{19, 20\}\},$

S5:  $\{\{1, 2\}, \{1, 4\}, \{1, 7\}, \{2, 3\}, \{2, 8\}, \{3, 5\}, \{3, 10\}, \{4, 9\}, \{4, 10\}, \{5, 6\}, \{5, 9\}, \{6, 7\}, \{6, 11\}, \{7, 19\}, \{8, 9\}, \{8, 12\}, \{10, 17\}, \{11, 12\}, \{11, 14\}, \{12, 13\}, \{13, 16\}, \{13, 18\}, \{14, 15\}, \{14, 18\}, \{15, 16\}, \{15, 17\}, \{16, 19\}, \{17, 20\}, \{18, 20\}, \{19, 20\}\}$  and

S6:  $\{\{1, 2\}, \{1, 4\}, \{1, 7\}, \{2, 3\}, \{2, 8\}, \{3, 5\}, \{3, 10\}, \{4, 9\}, \{4, 17\}, \{5, 6\}, \{5, 9\}, \{6, 7\}, \{6, 11\}, \{7, 19\}, \{8, 9\}, \{8, 12\}, \{10, 12\}, \{10, 13\}, \{11, 12\}, \{11, 14\}, \{13, 16\}, \{13, 18\}, \{14, 15\}, \{14, 18\}, \{15, 16\}, \{15, 17\}, \{16, 19\}, \{17, 20\}, \{18, 20\}, \{19, 20\}\}$

respectively. The first one is the flower snark J5 as supplied with the MATHEMATICA system, [24]. The remaining five are the same edges sets as appearing in [23] and [25, p. 276], where the graphs are called Sn5, Sn6, ..., Sn9 respectively. (We have not used the 20-vertex graph labelled Sn4 in [23].) Our graphs are pairwise isomorphic to the six 20-vertex snarks in Royale's list, [26], in order (6, 1, 5, 3, 2, 4).

**Lemma 4.1** *For each of the six snarks on 20 vertices, there exist designs of order 25, 40, 61, 76, 85, 121, 160, 181 and 220.*

**Proof.** Let the vertex set of  $K_{25}$  be  $Z_{25}$ . The decompositions consist of

$(16, 22, 10, 20, 18, 12, 3, 17, 11, 14, 6, 7, 0, 1, 19, 2, 4, 8, 5, 9)_{S1},$   
 $(0, 5, 1, 3, 9, 20, 13, 11, 23, 10, 16, 2, 14, 22, 24, 7, 19, 21, 8, 18)_{S1},$   
 $(22, 16, 8, 15, 2, 7, 3, 9, 0, 6, 21, 12, 1, 4, 5, 10, 13, 14, 11, 18)_{S2},$   
 $(0, 2, 7, 4, 9, 13, 23, 21, 1, 22, 5, 15, 3, 20, 18, 12, 19, 24, 16, 14)_{S2},$   
 $(1, 23, 2, 12, 15, 7, 17, 21, 16, 9, 19, 6, 0, 3, 4, 5, 8, 14, 11, 20)_{S3},$   
 $(0, 2, 1, 3, 9, 24, 4, 5, 16, 18, 15, 23, 17, 13, 22, 10, 8, 12, 6, 14)_{S3},$   
 $(11, 16, 19, 5, 14, 9, 3, 13, 15, 12, 23, 2, 0, 7, 1, 8, 10, 6, 4, 17)_{S4},$   
 $(0, 1, 2, 3, 10, 20, 17, 5, 24, 8, 23, 18, 7, 11, 4, 12, 19, 9, 21, 6)_{S4},$   
 $(17, 0, 10, 13, 21, 12, 23, 8, 3, 15, 9, 4, 1, 2, 5, 6, 11, 7, 14, 20)_{S5},$   
 $(0, 2, 1, 3, 23, 24, 9, 14, 10, 16, 22, 15, 4, 13, 19, 6, 8, 11, 17, 7)_{S5}$

and

$(22, 21, 11, 2, 5, 20, 9, 13, 4, 14, 23, 17, 1, 0, 7, 8, 10, 12, 3, 6)_{S6},$   
 $(0, 1, 3, 2, 20, 14, 16, 22, 13, 18, 24, 6, 5, 23, 19, 10, 12, 9, 21, 4)_{S6}$

under the action of the mapping  $x \mapsto x + 5 \pmod{25}$ .

Let the vertex set of  $K_{40}$  be  $Z_{39} \cup \{\infty\}$ . The decompositions consist of

$(\infty, 26, 24, 38, 28, 15, 2, 18, 34, 10, 19, 22, 13, 3, 8, 11, 6, 1, 4, 9)_{S1},$   
 $(0, 4, 2, 5, 9, 7, 11, 17, 21, 35, 24, 15, 38, 29, 1, 13, 16, 36, 26, 18)_{S1},$   
 $(\infty, 2, 1, 14, 24, 3, 9, 10, 38, 19, 11, 37, 20, 0, 4, 6, 12, 7, 5, 34)_{S2},$   
 $(0, 2, 7, 4, 8, 1, 12, 11, 22, 27, 24, 6, 33, 35, 14, 20, 37, 31, 3, 23)_{S2},$   
 $(\infty, 36, 30, 23, 34, 16, 7, 10, 14, 12, 26, 4, 19, 0, 1, 3, 6, 5, 2, 8)_{S3},$   
 $(0, 3, 7, 8, 1, 2, 9, 10, 30, 23, 13, 27, 12, 20, 22, 35, 19, 32, 21, 11)_{S3},$   
 $(\infty, 30, 37, 22, 17, 32, 5, 21, 31, 7, 13, 0, 24, 1, 2, 4, 6, 8, 3, 9)_{S4},$   
 $(0, 4, 5, 1, 3, 11, 14, 8, 15, 24, 28, 2, 9, 33, 19, 37, 29, 20, 22, 38)_{S4},$   
 $(\infty, 22, 2, 33, 23, 14, 20, 21, 35, 3, 6, 25, 15, 0, 5, 1, 10, 11, 4, 19)_{S5},$

$(0, 3, 1, 12, 2, 4, 15, 16, 5, 28, 11, 37, 23, 24, 19, 13, 36, 8, 35, 18)_{S5}$   
 and  
 $(\infty, 29, 32, 12, 9, 16, 4, 20, 27, 37, 3, 7, 35, 0, 1, 2, 6, 8, 10, 18)_{S6}$ ,  
 $(0, 2, 3, 5, 1, 4, 11, 6, 20, 12, 14, 32, 31, 37, 13, 17, 27, 21, 28, 10)_{S6}$   
 under the action of the mapping  $\infty \mapsto \infty, x \mapsto x + 3 \pmod{39}$ .  
 Let the vertex set of  $K_{61}$  be  $Z_{61}$ . The decompositions consist of  
 $(34, 21, 56, 58, 43, 46, 14, 41, 44, 51, 0, 28, 1, 3, 7, 19, 9, 40, 52, 29)_{S1}$ ,  
 $(38, 40, 42, 7, 0, 22, 6, 25, 48, 11, 29, 58, 2, 1, 4, 24, 56, 41, 14, 49)_{S2}$ ,  
 $(36, 56, 34, 15, 40, 22, 20, 10, 45, 26, 23, 59, 2, 4, 7, 16, 44, 8, 3, 37)_{S3}$ ,  
 $(5, 28, 19, 7, 24, 54, 57, 41, 8, 39, 59, 37, 2, 4, 23, 48, 50, 14, 56, 21)_{S4}$ ,  
 $(12, 54, 44, 56, 7, 34, 45, 39, 55, 30, 5, 9, 0, 2, 23, 41, 25, 38, 53, 31)_{S5}$   
 and  
 $(3, 47, 29, 2, 34, 24, 16, 28, 43, 32, 12, 57, 1, 5, 11, 35, 13, 27, 39, 60)_{S6}$   
 under the action of the mapping  $x \mapsto x + 1 \pmod{61}$ .  
 Let the vertex set of  $K_{76}$  be  $Z_{76}$ . The decompositions consist of  
 $(21, 69, 20, 4, 7, 63, 67, 70, 30, 18, 14, 59, 38, 48, 9, 0, 51, 3, 47, 6)_{S1}$ ,  
 $(29, 66, 50, 33, 73, 26, 53, 25, 71, 52, 67, 60, 49, 54, 44, 56, 72, 21, 15, 9)_{S1}$ ,  
 $(41, 3, 27, 43, 1, 15, 6, 26, 19, 13, 5, 12, 28, 57, 70, 60, 30, 66, 20, 11)_{S1}$ ,  
 $(26, 58, 63, 6, 33, 57, 44, 61, 16, 36, 31, 2, 43, 74, 21, 52, 55, 4, 0, 30)_{S1}$ ,  
 $(59, 10, 32, 38, 60, 67, 8, 64, 45, 61, 52, 39, 35, 9, 54, 70, 25, 56, 19, 37)_{S1}$ ,  
 $(3, 14, 1, 55, 7, 15, 60, 49, 21, 4, 42, 26, 28, 61, 58, 69, 62, 47, 40, 54)_{S2}$ ,  
 $(20, 24, 69, 23, 9, 13, 4, 1, 37, 5, 54, 11, 68, 28, 48, 70, 18, 12, 21, 62)_{S2}$ ,  
 $(2, 24, 25, 15, 27, 63, 48, 20, 49, 6, 50, 74, 36, 0, 47, 46, 5, 23, 42, 54)_{S2}$ ,  
 $(12, 48, 23, 42, 60, 8, 52, 63, 1, 37, 32, 59, 71, 51, 7, 15, 22, 6, 14, 5)_{S2}$ ,  
 $(44, 62, 57, 59, 19, 43, 60, 46, 36, 75, 33, 13, 37, 67, 1, 22, 71, 29, 53, 66)_{S2}$ ,  
 $(32, 63, 49, 62, 19, 26, 65, 47, 30, 48, 10, 53, 12, 14, 67, 29, 57, 50, 11, 5)_{S3}$ ,  
 $(60, 7, 70, 73, 42, 72, 12, 29, 32, 18, 51, 40, 25, 14, 13, 6, 45, 48, 64, 50)_{S3}$ ,  
 $(34, 60, 55, 28, 70, 35, 36, 69, 21, 64, 72, 33, 44, 31, 41, 59, 53, 19, 56, 50)_{S3}$ ,  
 $(46, 38, 50, 37, 20, 16, 24, 25, 67, 71, 60, 9, 10, 45, 74, 21, 59, 43, 6, 63)_{S3}$ ,  
 $(65, 69, 19, 8, 7, 34, 35, 49, 20, 27, 59, 0, 55, 30, 10, 15, 45, 58, 67, 24)_{S3}$ ,  
 $(46, 13, 64, 4, 33, 26, 5, 14, 19, 2, 7, 34, 48, 28, 40, 18, 66, 25, 37, 27)_{S4}$ ,  
 $(14, 53, 41, 12, 48, 61, 45, 16, 72, 71, 66, 18, 44, 57, 23, 38, 11, 67, 74, 63)_{S4}$ ,  
 $(24, 59, 39, 46, 38, 71, 45, 51, 22, 0, 23, 5, 19, 41, 58, 36, 48, 63, 7, 49)_{S4}$ ,  
 $(73, 61, 70, 36, 4, 53, 45, 5, 64, 9, 66, 16, 3, 14, 65, 25, 32, 59, 21, 19)_{S4}$ ,  
 $(23, 49, 39, 55, 64, 36, 28, 68, 30, 72, 54, 3, 57, 42, 35, 34, 14, 46, 31, 56)_{S4}$ ,  
 $(10, 23, 52, 32, 62, 60, 24, 37, 9, 0, 49, 50, 5, 56, 74, 51, 19, 25, 66, 20)_{S5}$ ,  
 $(60, 23, 68, 45, 69, 26, 71, 33, 63, 72, 12, 25, 64, 50, 7, 55, 54, 52, 30, 46)_{S5}$ ,  
 $(72, 56, 15, 22, 49, 7, 3, 43, 17, 42, 2, 6, 54, 57, 0, 55, 66, 24, 63, 41)_{S5}$ ,  
 $(50, 69, 2, 21, 61, 54, 13, 65, 71, 72, 20, 49, 73, 0, 28, 55, 37, 51, 74, 34)_{S5}$ ,

(15, 0, 21, 71, 24, 16, 59, 50, 47, 35, 11, 28, 31, 22, 10, 19, 42, 33, 57, 69)<sub>S5</sub>

and

(12, 60, 70, 56, 45, 18, 55, 23, 17, 13, 53, 32, 67, 35, 24, 64, 25, 41, 58, 61)<sub>S6</sub>,

(19, 3, 71, 57, 20, 2, 10, 27, 72, 8, 1, 14, 15, 24, 7, 5, 34, 49, 12, 67)<sub>S6</sub>,

(14, 7, 22, 60, 74, 8, 28, 55, 34, 5, 12, 26, 1, 72, 64, 61, 3, 13, 69, 45)<sub>S6</sub>,

(20, 55, 8, 29, 62, 18, 22, 57, 24, 30, 49, 10, 21, 60, 73, 43, 59, 26, 3, 14)<sub>S6</sub>,

(36, 70, 1, 59, 48, 43, 63, 10, 60, 66, 11, 15, 18, 23, 49, 0, 6, 33, 46, 47)<sub>S6</sub>

under the action of the mapping  $x \mapsto x + 4 \pmod{76}$ .

Let the vertex set of  $K_{85}$  be  $Z_{85}$ . The decompositions consist of

(13, 40, 57, 34, 24, 1, 14, 16, 10, 6, 64, 27, 62, 52, 46, 69, 59, 77, 81, 4)<sub>S1</sub>,

(43, 74, 47, 64, 21, 49, 34, 81, 39, 13, 6, 9, 56, 32, 73, 68, 75, 26, 62, 42)<sub>S1</sub>,

(15, 17, 82, 2, 53, 18, 27, 32, 26, 52, 61, 30, 71, 58, 55, 11, 28, 9, 20, 45)<sub>S1</sub>,

(42, 15, 80, 19, 12, 51, 40, 10, 5, 26, 83, 46, 67, 34, 48, 62, 4, 65, 70, 37)<sub>S1</sub>,

(21, 22, 18, 64, 5, 83, 27, 38, 3, 42, 43, 80, 28, 79, 81, 76, 46, 75, 8, 59)<sub>S1</sub>,

(47, 28, 50, 12, 7, 80, 38, 24, 14, 66, 15, 5, 22, 35, 4, 65, 36, 41, 73, 11)<sub>S1</sub>,

(71, 17, 79, 68, 0, 23, 59, 29, 44, 43, 53, 51, 84, 65, 48, 25, 45, 8, 56, 63)<sub>S1</sub>,

(41, 83, 25, 65, 75, 13, 69, 29, 6, 53, 40, 51, 45, 64, 47, 60, 63, 77, 72, 46)<sub>S2</sub>,

(83, 8, 16, 74, 48, 17, 42, 47, 6, 38, 35, 26, 23, 14, 66, 20, 82, 39, 11, 19)<sub>S2</sub>,

(36, 66, 1, 0, 40, 73, 80, 37, 2, 39, 30, 83, 46, 47, 81, 9, 52, 79, 45, 10)<sub>S2</sub>,

(37, 73, 61, 75, 51, 58, 18, 20, 72, 33, 46, 27, 7, 66, 77, 63, 64, 0, 54, 4)<sub>S2</sub>,

(13, 47, 39, 77, 49, 82, 70, 53, 80, 72, 59, 27, 31, 76, 5, 14, 61, 40, 16, 17)<sub>S2</sub>,

(31, 75, 38, 47, 10, 56, 77, 24, 42, 63, 64, 57, 13, 51, 12, 2, 54, 69, 19, 14)<sub>S2</sub>,

(36, 10, 26, 30, 64, 55, 63, 35, 83, 69, 48, 53, 38, 14, 58, 49, 71, 28, 75, 44)<sub>S2</sub>,

(77, 41, 6, 42, 1, 53, 28, 33, 46, 5, 19, 47, 12, 32, 56, 37, 60, 15, 35, 78)<sub>S3</sub>,

(60, 7, 29, 65, 53, 55, 80, 72, 69, 77, 34, 67, 63, 49, 6, 82, 64, 44, 48, 8)<sub>S3</sub>,

(18, 56, 13, 24, 15, 76, 9, 4, 60, 54, 73, 65, 34, 35, 19, 36, 31, 74, 67, 21)<sub>S3</sub>,

(16, 76, 21, 1, 13, 59, 81, 39, 70, 14, 28, 32, 41, 29, 43, 73, 22, 49, 3, 53)<sub>S3</sub>,

(75, 78, 76, 42, 21, 44, 12, 56, 50, 82, 10, 83, 35, 72, 31, 20, 47, 34, 3, 23)<sub>S3</sub>,

(83, 60, 5, 46, 50, 23, 67, 41, 0, 12, 6, 18, 16, 70, 29, 19, 53, 11, 79, 20)<sub>S3</sub>,

(39, 22, 36, 13, 47, 32, 45, 77, 18, 0, 24, 10, 21, 15, 72, 49, 74, 42, 7, 53)<sub>S3</sub>,

(38, 42, 23, 61, 7, 46, 63, 44, 75, 0, 25, 4, 10, 60, 13, 59, 52, 47, 65, 20)<sub>S4</sub>,

(51, 70, 80, 54, 77, 2, 10, 52, 81, 15, 74, 75, 83, 62, 17, 12, 41, 82, 63, 34)<sub>S4</sub>,

(76, 32, 82, 38, 36, 78, 23, 4, 63, 73, 43, 53, 6, 3, 11, 81, 61, 27, 19, 65)<sub>S4</sub>,

(52, 67, 30, 32, 46, 41, 19, 4, 6, 35, 80, 54, 79, 66, 50, 78, 57, 11, 34, 27)<sub>S4</sub>,

(72, 4, 54, 80, 69, 26, 46, 62, 3, 40, 35, 83, 43, 48, 77, 25, 65, 70, 13, 16)<sub>S4</sub>,

(62, 20, 59, 22, 38, 29, 5, 31, 49, 28, 24, 4, 82, 3, 77, 33, 69, 81, 39, 53)<sub>S4</sub>,

(50, 20, 76, 28, 6, 44, 56, 71, 39, 42, 14, 74, 31, 70, 63, 68, 16, 67, 81, 24)<sub>S4</sub>,

(63, 74, 46, 58, 52, 66, 64, 43, 10, 11, 41, 24, 32, 42, 9, 2, 57, 59, 13, 0)<sub>S5</sub>,

(58, 51, 14, 73, 77, 39, 1, 43, 57, 10, 63, 72, 20, 53, 21, 12, 3, 30, 27, 15)<sub>S5</sub>,

(1, 41, 7, 23, 0, 20, 31, 5, 6, 2, 9, 54, 61, 80, 14, 29, 66, 56, 45, 43)<sub>S5</sub>,

(3, 27, 80, 28, 29, 4, 1, 81, 19, 61, 17, 77, 60, 13, 40, 10, 42, 56, 37, 83)<sub>S5</sub>,

(41, 49, 7, 12, 52, 58, 75, 47, 36, 38, 19, 37, 9, 42, 65, 73, 34, 23, 8, 64)<sub>S5</sub>,  
 (46, 24, 18, 25, 63, 35, 75, 73, 4, 68, 26, 70, 38, 44, 20, 2, 67, 0, 50, 79)<sub>S5</sub>,  
 (28, 58, 7, 31, 10, 49, 76, 46, 15, 70, 36, 56, 14, 21, 80, 67, 79, 9, 32, 29)<sub>S5</sub>

and

(59, 51, 38, 17, 78, 5, 9, 68, 62, 37, 11, 45, 40, 64, 72, 71, 49, 33, 69, 8)<sub>S6</sub>,  
 (38, 29, 23, 64, 9, 21, 56, 19, 82, 27, 68, 25, 10, 15, 62, 49, 30, 72, 22, 6)<sub>S6</sub>,  
 (16, 7, 4, 9, 59, 41, 81, 36, 50, 15, 47, 61, 44, 35, 77, 23, 66, 80, 39, 70)<sub>S6</sub>,  
 (48, 75, 42, 63, 67, 62, 29, 82, 52, 22, 36, 32, 61, 39, 60, 74, 80, 66, 18, 55)<sub>S6</sub>,  
 (7, 64, 19, 44, 12, 3, 58, 68, 61, 2, 4, 66, 50, 20, 21, 48, 6, 41, 77, 38)<sub>S6</sub>,  
 (36, 77, 53, 66, 17, 58, 48, 18, 38, 67, 6, 69, 80, 39, 76, 22, 50, 54, 40, 55)<sub>S6</sub>,  
 (65, 10, 32, 68, 56, 48, 30, 38, 33, 79, 9, 74, 55, 75, 53, 11, 31, 3, 1, 50)<sub>S6</sub>

under the action of the mapping  $x \mapsto x + 5 \pmod{85}$ .

Let the vertex set of  $K_{121}$  be  $Z_{121}$ . The decompositions consist of

(41, 27, 34, 87, 98, 81, 0, 92, 73, 2,  
 118, 24, 1, 3, 4, 6, 5, 11, 19, 46)<sub>S1</sub>,  
 (0, 10, 25, 26, 18, 37, 5, 64, 41, 3,  
 31, 34, 4, 48, 114, 1, 73, 6, 71, 68)<sub>S1</sub>,  
 (76, 21, 82, 29, 15, 33, 55, 41, 36, 75,  
 49, 100, 0, 1, 2, 5, 4, 14, 11, 27)<sub>S2</sub>,  
 (0, 14, 15, 31, 18, 1, 38, 56, 2, 80,  
 41, 74, 3, 49, 107, 45, 76, 89, 78, 9)<sub>S2</sub>,  
 (79, 80, 5, 9, 56, 97, 64, 90, 18, 78,  
 1, 35, 91, 37, 30, 44, 47, 41, 75, 73)<sub>S3</sub>,  
 (104, 74, 29, 46, 45, 67, 64, 111, 58, 90,  
 47, 106, 6, 66, 42, 70, 50, 12, 41, 68)<sub>S3</sub>,  
 (78, 119, 72, 76, 11, 69, 82, 32, 39, 61,  
 47, 116, 0, 1, 2, 5, 4, 9, 15, 27)<sub>S4</sub>,  
 (0, 14, 15, 30, 1, 18, 38, 33, 54, 39,  
 43, 3, 4, 17, 93, 46, 120, 81, 97, 49)<sub>S4</sub>,  
 (32, 14, 24, 92, 86, 40, 81, 34, 111, 78,  
 11, 104, 0, 2, 1, 3, 5, 7, 15, 28)<sub>S5</sub>,  
 (0, 6, 14, 11, 29, 2, 24, 22, 61, 44,  
 28, 59, 7, 114, 33, 75, 91, 71, 109, 26)<sub>S5</sub>

and

(95, 69, 98, 15, 104, 103, 118, 41, 39, 114,  
 48, 66, 0, 1, 4, 8, 24, 11, 27, 60)<sub>S6</sub>,  
 (0, 5, 17, 13, 3, 20, 42, 26, 53, 48,  
 52, 87, 107, 105, 18, 56, 70, 62, 100, 16)<sub>S6</sub>

under the action of the mapping  $x \mapsto x + 1 \pmod{121}$ .

Let the vertex set of  $K_{160}$  be  $Z_{159} \cup \{\infty\}$ . The decompositions consist of



$(\infty, 65, 114, 84, 36, 18, 151, 19, 88, 96,$   
 $90, 98, 7, 4, 158, 44, 94, 25, 54, 63)_{S1},$   
 $(46, 30, 155, 142, 1, 24, 118, 58, 106, 41,$   
 $59, 83, 62, 47, 80, 6, 101, 123, 34, 66)_{S1},$   
 $(158, 154, 77, 13, 141, 149, 71, 28, 42, 84,$   
 $69, 34, 64, 75, 48, 78, 31, 1, 104, 113)_{S1},$   
 $(113, 141, 22, 140, 46, 103, 29, 92, 146, 136,$   
 $9, 117, 122, 3, 13, 34, 142, 116, 86, 28)_{S1},$   
 $(36, 158, 15, 94, 139, 112, 21, 119, 151, 35,$   
 $42, 149, 5, 153, 74, 132, 117, 69, 45, 2)_{S1},$   
 $(17, 100, 14, 16, 145, 22, 41, 57, 2, 59,$   
 $86, 3, 43, 126, 82, 51, 26, 99, 112, 104)_{S1},$   
 $(56, 5, 19, 68, 81, 131, 72, 24, 149, 137,$   
 $44, 116, 143, 145, 7, 141, 64, 25, 45, 53)_{S1},$   
 $(149, 75, 112, 120, 13, 73, 7, 121, 129, 2,$   
 $63, 60, 3, 10, 72, 27, 88, 147, 81, 70)_{S1},$   
 $(\infty, 128, 3, 104, 16, 81, 113, 157, 133, 72,$   
 $41, 74, 105, 141, 156, 9, 111, 84, 122, 78)_{S2},$   
 $(96, 28, 44, 13, 85, 120, 128, 21, 97, 22,$   
 $76, 67, 91, 69, 155, 84, 102, 115, 108, 5)_{S2},$   
 $(111, 58, 18, 46, 53, 97, 62, 136, 106, 34,$   
 $79, 112, 114, 133, 57, 76, 66, 13, 3, 29)_{S2},$   
 $(65, 82, 54, 32, 156, 99, 69, 62, 55, 74,$   
 $105, 83, 149, 141, 20, 128, 129, 18, 76, 53)_{S2},$   
 $(156, 158, 79, 145, 122, 103, 137, 149, 59, 0,$   
 $84, 113, 33, 6, 104, 151, 37, 42, 138, 142)_{S2},$   
 $(125, 7, 115, 14, 73, 62, 108, 39, 37, 11,$   
 $44, 118, 15, 56, 155, 158, 52, 0, 88, 77)_{S2},$   
 $(87, 67, 97, 70, 112, 3, 17, 42, 106, 29,$   
 $18, 40, 35, 21, 125, 30, 11, 51, 148, 109)_{S2},$   
 $(69, 126, 109, 129, 112, 125, 139, 11, 101, 54,$   
 $86, 7, 67, 62, 20, 38, 143, 107, 22, 23)_{S2},$   
 $(\infty, 18, 107, 31, 28, 87, 80, 38, 129, 88,$   
 $158, 116, 127, 8, 69, 2, 19, 114, 144, 103)_{S3},$   
 $(31, 123, 60, 67, 93, 152, 68, 90, 89, 85,$   
 $144, 139, 76, 11, 77, 96, 153, 146, 141, 156)_{S3},$   
 $(13, 27, 11, 79, 135, 66, 94, 139, 63, 6,$   
 $53, 140, 30, 145, 97, 44, 7, 131, 74, 92)_{S3},$   
 $(19, 71, 133, 82, 104, 64, 14, 39, 30, 103,$   
 $52, 12, 69, 11, 32, 81, 114, 115, 59, 124)_{S3},$   
 $(26, 124, 97, 78, 62, 156, 34, 47, 85, 140,$   
 $75, 93, 56, 54, 136, 112, 103, 139, 125, 92)_{S3},$

$(14, 48, 128, 110, 131, 71, 40, 118, 8, 125,$   
 $23, 157, 27, 11, 13, 57, 77, 91, 86, 117)_{S_3},$   
 $(37, 39, 7, 3, 82, 27, 75, 141, 133, 14,$   
 $70, 135, 140, 85, 130, 86, 6, 138, 114, 30)_{S_3},$   
 $(68, 85, 108, 111, 37, 124, 45, 56, 40, 152,$   
 $120, 83, 110, 58, 114, 54, 150, 100, 76, 41)_{S_3},$   
 $(\infty, 44, 151, 13, 132, 101, 33, 5, 86, 21,$   
 $134, 136, 6, 4, 90, 18, 73, 53, 108, 119)_{S_4},$   
 $(73, 36, 100, 124, 66, 41, 9, 121, 93, 37,$   
 $75, 32, 31, 141, 68, 76, 20, 115, 145, 154)_{S_4},$   
 $(1, 53, 104, 70, 57, 16, 93, 33, 39, 134,$   
 $109, 12, 72, 157, 5, 63, 124, 19, 88, 8)_{S_4},$   
 $(56, 128, 111, 91, 155, 101, 127, 84, 5, 72,$   
 $43, 110, 115, 143, 158, 85, 124, 133, 4, 12)_{S_4},$   
 $(136, 89, 105, 112, 134, 155, 28, 26, 150, 44,$   
 $152, 3, 63, 77, 90, 119, 14, 41, 60, 59)_{S_4},$   
 $(158, 116, 52, 19, 123, 41, 82, 59, 66, 68,$   
 $69, 119, 44, 65, 38, 88, 34, 79, 14, 0)_{S_4},$   
 $(74, 112, 123, 55, 68, 156, 5, 42, 145, 121,$   
 $71, 141, 120, 65, 46, 57, 114, 53, 3, 36)_{S_4},$   
 $(150, 21, 108, 97, 93, 117, 153, 31, 43, 127,$   
 $9, 55, 130, 133, 72, 88, 77, 81, 89, 129)_{S_4},$   
 $(\infty, 146, 93, 138, 120, 123, 106, 108, 25, 67,$   
 $156, 40, 139, 87, 47, 64, 97, 137, 98, 82)_{S_5},$   
 $(54, 45, 117, 50, 134, 82, 89, 90, 145, 17,$   
 $35, 69, 12, 132, 15, 151, 113, 7, 157, 44)_{S_5},$   
 $(3, 47, 15, 118, 37, 17, 154, 106, 73, 122,$   
 $109, 55, 46, 7, 36, 41, 66, 147, 151, 74)_{S_5},$   
 $(58, 35, 63, 100, 91, 147, 14, 48, 87, 15,$   
 $19, 5, 151, 115, 140, 68, 82, 62, 45, 92)_{S_5},$   
 $(145, 51, 26, 133, 74, 47, 127, 61, 35, 108,$   
 $5, 123, 24, 121, 69, 18, 33, 89, 0, 14)_{S_5},$   
 $(129, 118, 111, 19, 18, 92, 158, 91, 122, 113,$   
 $4, 52, 142, 102, 55, 56, 114, 64, 11, 151)_{S_5},$   
 $(25, 65, 132, 110, 43, 22, 24, 106, 120, 82,$   
 $157, 141, 87, 91, 56, 41, 135, 29, 119, 128)_{S_5},$   
 $(141, 117, 157, 104, 128, 144, 129, 54, 125, 84,$   
 $95, 133, 15, 44, 50, 26, 14, 98, 48, 116)_{S_5}$   
 and  
 $(\infty, 142, 111, 29, 46, 140, 15, 129, 105, 106,$   
 $148, 85, 67, 61, 80, 7, 95, 150, 54, 16)_{S_6},$

(72, 130, 55, 23, 28, 85, 109, 156, 43, 116,  
 30, 102, 104, 146, 8, 66, 135, 101, 144, 38)<sub>S6</sub>,  
 (44, 134, 66, 132, 85, 63, 7, 16, 113, 118,  
 23, 10, 158, 102, 74, 81, 64, 128, 110, 68)<sub>S6</sub>,  
 (92, 65, 34, 140, 32, 89, 90, 79, 150, 111,  
 25, 91, 100, 77, 28, 145, 104, 136, 48, 137)<sub>S6</sub>,  
 (117, 125, 154, 25, 64, 82, 33, 13, 80, 73,  
 59, 35, 43, 68, 103, 86, 136, 34, 40, 6)<sub>S6</sub>,  
 (0, 30, 65, 9, 42, 121, 16, 90, 24, 81,  
 44, 138, 126, 115, 123, 143, 82, 105, 48, 99)<sub>S6</sub>,  
 (145, 0, 146, 144, 39, 89, 95, 33, 44, 40,  
 157, 125, 135, 96, 46, 99, 48, 139, 14, 134)<sub>S6</sub>,  
 (149, 124, 127, 91, 101, 45, 8, 131, 47, 150,  
 158, 3, 76, 21, 48, 146, 51, 65, 95, 120)<sub>S6</sub>  
 under the action of the mapping  $\infty \mapsto \infty$ ,  $x \mapsto x + 3 \pmod{159}$ .  
 Let the vertex set of  $K_{181}$  be  $Z_{181}$ . The decompositions consist of  
 (98, 69, 51, 61, 3, 133, 64, 4, 82, 132,  
 23, 164, 91, 74, 49, 67, 56, 34, 104, 10)<sub>S1</sub>,  
 (43, 81, 127, 6, 57, 175, 22, 74, 37, 178,  
 18, 7, 71, 167, 161, 154, 148, 16, 100, 104)<sub>S1</sub>,  
 (15, 118, 58, 44, 132, 50, 48, 123, 121, 133,  
 0, 43, 23, 53, 114, 145, 82, 51, 77, 113)<sub>S1</sub>,  
 (114, 15, 165, 106, 100, 85, 94, 67, 93, 55,  
 87, 30, 115, 131, 10, 54, 112, 18, 45, 107)<sub>S2</sub>,  
 (109, 89, 116, 102, 3, 57, 169, 144, 173, 114,  
 62, 136, 119, 25, 78, 145, 59, 32, 150, 67)<sub>S2</sub>,  
 (77, 59, 53, 139, 54, 111, 70, 5, 80, 134,  
 1, 30, 8, 47, 155, 112, 79, 101, 102, 167)<sub>S2</sub>,  
 (146, 52, 106, 104, 150, 117, 45, 12, 162, 30,  
 4, 1, 177, 87, 9, 57, 55, 173, 38, 56)<sub>S3</sub>,  
 (6, 110, 72, 118, 12, 127, 161, 116, 69, 7,  
 97, 129, 170, 60, 101, 51, 80, 45, 35, 55)<sub>S3</sub>,  
 (149, 97, 102, 65, 73, 82, 60, 121, 164, 63,  
 1, 46, 6, 64, 137, 77, 0, 99, 113, 85)<sub>S3</sub>,  
 (97, 25, 150, 14, 33, 122, 74, 87, 62, 20,  
 27, 128, 24, 171, 54, 148, 136, 105, 110, 103)<sub>S4</sub>,  
 (74, 86, 40, 164, 48, 80, 134, 30, 58, 119,  
 176, 141, 114, 56, 98, 67, 101, 1, 91, 85)<sub>S4</sub>,  
 (35, 26, 172, 66, 170, 16, 74, 161, 6, 5,  
 3, 55, 4, 154, 63, 177, 137, 109, 157, 88)<sub>S4</sub>,

(151, 157, 161, 162, 176, 24, 106, 8, 171, 42,  
 38, 11, 81, 126, 175, 120, 121, 9, 25, 77)<sub>S5</sub>,  
 (31, 82, 139, 125, 72, 179, 156, 130, 22, 137,  
 5, 95, 70, 52, 98, 167, 9, 44, 16, 166)<sub>S5</sub>,  
 (36, 107, 30, 58, 11, 164, 19, 65, 45, 68,  
 4, 5, 80, 37, 100, 176, 166, 138, 160, 22)<sub>S5</sub>

and

(46, 111, 108, 22, 161, 90, 102, 157, 114, 23,  
 14, 93, 179, 133, 170, 79, 129, 178, 47, 20)<sub>S6</sub>,  
 (108, 77, 178, 112, 11, 164, 110, 140, 18, 145,  
 61, 160, 12, 70, 51, 85, 134, 176, 74, 124)<sub>S6</sub>,  
 (168, 36, 42, 130, 93, 16, 100, 175, 7, 58,  
 8, 29, 14, 77, 51, 165, 90, 138, 160, 72)<sub>S6</sub>

under the action of the mapping  $x \mapsto x + 1 \pmod{181}$ .

Let the vertex set of  $K_{220}$  be  $Z_{219} \cup \{\infty\}$ . The decompositions consist of

( $\infty$ , 69, 185, 32, 110, 27, 167, 122, 196, 123,  
 7, 12, 159, 5, 34, 86, 67, 207, 1, 105)<sub>S1</sub>,  
 (58, 59, 15, 207, 17, 35, 40, 49, 14, 186,  
 61, 184, 115, 135, 122, 6, 159, 199, 127, 26)<sub>S1</sub>,  
 (40, 34, 30, 91, 57, 51, 53, 100, 36, 96,  
 149, 195, 179, 180, 6, 205, 58, 123, 124, 115)<sub>S1</sub>,  
 (40, 205, 141, 181, 4, 124, 74, 82, 59, 106,  
 103, 201, 93, 136, 97, 150, 109, 197, 69, 13)<sub>S1</sub>,  
 (193, 85, 101, 137, 52, 50, 30, 179, 155, 81,  
 26, 203, 131, 45, 127, 9, 17, 32, 0, 178)<sub>S1</sub>,  
 (37, 126, 40, 184, 163, 148, 23, 53, 49, 123,  
 5, 67, 194, 96, 72, 13, 188, 121, 61, 159)<sub>S1</sub>,  
 (68, 110, 190, 80, 101, 133, 20, 62, 66, 37,  
 48, 10, 116, 201, 140, 0, 14, 158, 89, 58)<sub>S1</sub>,  
 (204, 146, 33, 119, 154, 52, 129, 75, 137, 115,  
 23, 205, 125, 102, 177, 216, 176, 79, 26, 14)<sub>S1</sub>,  
 (49, 171, 109, 156, 53, 60, 123, 21, 215, 36,  
 9, 185, 176, 140, 76, 160, 56, 150, 134, 146)<sub>S1</sub>,  
 (173, 194, 189, 166, 65, 76, 10, 116, 197, 37,  
 6, 63, 159, 18, 186, 171, 128, 109, 99, 154)<sub>S1</sub>,  
 (33, 8, 193, 75, 164, 28, 201, 186, 96, 93,  
 24, 152, 202, 44, 144, 159, 211, 188, 116, 163)<sub>S1</sub>,  
 ( $\infty$ , 183, 191, 105, 193, 20, 99, 107, 181, 120,  
 165, 153, 128, 82, 15, 204, 73, 9, 45, 60)<sub>S2</sub>,  
 (135, 24, 177, 92, 204, 25, 39, 138, 68, 86,  
 188, 139, 122, 170, 190, 9, 155, 110, 19, 30)<sub>S2</sub>,

(76, 64, 192, 109, 169, 65, 23, 37, 194, 106,  
 146, 70, 201, 97, 55, 139, 199, 43, 151, 203)<sub>S2</sub>,  
 (197, 78, 171, 175, 85, 176, 74, 205, 36, 39,  
 193, 7, 174, 111, 122, 61, 102, 151, 207, 13)<sub>S2</sub>,  
 (13, 169, 8, 60, 10, 139, 186, 124, 218, 149,  
 56, 215, 136, 20, 71, 100, 201, 157, 154, 180)<sub>S2</sub>,  
 (158, 189, 72, 121, 173, 43, 150, 10, 35, 56,  
 208, 98, 48, 137, 163, 65, 153, 193, 107, 54)<sub>S2</sub>,  
 (199, 20, 8, 28, 110, 196, 11, 164, 133, 9,  
 124, 148, 203, 190, 88, 119, 39, 3, 45, 187)<sub>S2</sub>,  
 (90, 171, 7, 14, 50, 177, 107, 28, 91, 26,  
 44, 82, 138, 172, 17, 9, 197, 48, 131, 5)<sub>S2</sub>,  
 (108, 45, 102, 72, 20, 167, 95, 194, 85, 98,  
 35, 112, 114, 59, 146, 155, 140, 111, 78, 213)<sub>S2</sub>,  
 (68, 158, 104, 115, 160, 147, 42, 181, 203, 6,  
 208, 168, 49, 169, 96, 48, 47, 185, 15, 75)<sub>S2</sub>,  
 (6, 13, 132, 21, 63, 66, 138, 67, 154, 115,  
 185, 161, 43, 80, 68, 102, 148, 179, 103, 72)<sub>S2</sub>,  
 ( $\infty$ , 125, 71, 123, 131, 43, 61, 36, 5, 148,  
 167, 140, 35, 68, 178, 171, 117, 173, 113, 105)<sub>S3</sub>,  
 (22, 24, 194, 178, 137, 204, 51, 73, 1, 43,  
 145, 74, 160, 52, 77, 32, 60, 66, 188, 101)<sub>S3</sub>,  
 (40, 23, 185, 163, 132, 188, 202, 60, 52, 119,  
 138, 152, 37, 50, 21, 142, 109, 2, 164, 182)<sub>S3</sub>,  
 (89, 198, 37, 78, 62, 26, 187, 200, 195, 138,  
 176, 9, 113, 116, 119, 6, 174, 178, 160, 8)<sub>S3</sub>,  
 (1, 87, 209, 184, 169, 82, 78, 133, 33, 199,  
 130, 121, 53, 137, 18, 44, 69, 187, 208, 5)<sub>S3</sub>,  
 (153, 146, 189, 1, 150, 110, 183, 21, 169, 151,  
 190, 130, 46, 24, 127, 121, 174, 20, 80, 0)<sub>S3</sub>,  
 (34, 210, 19, 95, 66, 39, 214, 28, 61, 206,  
 60, 216, 16, 159, 33, 211, 141, 78, 12, 183)<sub>S3</sub>,  
 (204, 179, 81, 50, 144, 109, 17, 36, 120, 65,  
 154, 37, 118, 47, 43, 137, 172, 131, 215, 87)<sub>S3</sub>,  
 (5, 160, 138, 11, 32, 174, 25, 130, 133, 151,  
 34, 141, 37, 115, 29, 202, 178, 26, 23, 79)<sub>S3</sub>,  
 (189, 9, 47, 155, 63, 72, 195, 25, 125, 108,  
 136, 107, 115, 70, 113, 41, 137, 11, 218, 159)<sub>S3</sub>,  
 (39, 184, 21, 3, 60, 138, 115, 150, 204, 90,  
 171, 58, 41, 99, 173, 185, 78, 154, 68, 81)<sub>S3</sub>,  
 ( $\infty$ , 16, 141, 118, 209, 8, 206, 197, 202, 211,  
 34, 105, 160, 69, 199, 100, 117, 46, 84, 72)<sub>S4</sub>,

(14, 21, 159, 179, 40, 37, 99, 164, 112, 103,  
 106, 140, 197, 151, 44, 107, 180, 51, 1, 163)<sub>S4</sub>,  
 (171, 9, 84, 95, 2, 137, 190, 153, 101, 131,  
 178, 14, 0, 55, 189, 108, 72, 38, 141, 192)<sub>S4</sub>,  
 (68, 33, 110, 14, 12, 141, 38, 186, 147, 1,  
 78, 218, 17, 187, 121, 103, 197, 29, 27, 130)<sub>S4</sub>,  
 (50, 211, 23, 216, 121, 75, 60, 152, 186, 118,  
 44, 104, 76, 101, 100, 9, 170, 105, 82, 57)<sub>S4</sub>,  
 (13, 154, 142, 90, 204, 100, 63, 18, 188, 110,  
 70, 7, 153, 103, 176, 99, 216, 53, 8, 130)<sub>S4</sub>,  
 (137, 181, 106, 43, 175, 189, 39, 50, 146, 130,  
 199, 207, 78, 110, 38, 157, 170, 138, 92, 125)<sub>S4</sub>,  
 (172, 89, 116, 21, 55, 76, 2, 194, 180, 107,  
 96, 174, 66, 58, 166, 176, 168, 97, 150, 171)<sub>S4</sub>,  
 (91, 176, 170, 23, 77, 181, 127, 98, 69, 20,  
 44, 55, 45, 66, 25, 182, 19, 39, 34, 46)<sub>S4</sub>,  
 (17, 197, 4, 192, 23, 43, 68, 194, 135, 84,  
 28, 179, 25, 163, 12, 201, 70, 216, 24, 15)<sub>S4</sub>,  
 (78, 114, 201, 70, 79, 107, 182, 118, 71, 22,  
 188, 166, 123, 67, 159, 65, 89, 99, 176, 112)<sub>S4</sub>,  
 ( $\infty$ , 2, 129, 162, 83, 82, 40, 84, 146, 109,  
 216, 211, 184, 77, 150, 200, 159, 17, 113, 164)<sub>S5</sub>,  
 (114, 45, 100, 84, 47, 115, 17, 1, 39, 193,  
 87, 104, 153, 31, 67, 207, 205, 199, 172, 212)<sub>S5</sub>,  
 (212, 147, 54, 25, 125, 136, 105, 120, 19, 143,  
 201, 77, 82, 110, 180, 84, 113, 112, 121, 199)<sub>S5</sub>,  
 (186, 157, 81, 187, 140, 21, 44, 40, 162, 116,  
 193, 103, 182, 129, 151, 108, 79, 143, 210, 2)<sub>S5</sub>,  
 (109, 50, 90, 185, 62, 79, 29, 154, 211, 128,  
 64, 87, 2, 119, 153, 121, 134, 195, 30, 124)<sub>S5</sub>,  
 (29, 98, 70, 83, 57, 63, 73, 197, 16, 128,  
 27, 30, 116, 174, 91, 182, 152, 23, 4, 58)<sub>S5</sub>,  
 (69, 7, 181, 170, 92, 74, 189, 206, 123, 173,  
 158, 216, 204, 171, 153, 149, 15, 114, 101, 148)<sub>S5</sub>,  
 (14, 12, 52, 70, 134, 114, 51, 68, 46, 78,  
 155, 151, 200, 32, 55, 89, 162, 215, 183, 88)<sub>S5</sub>,  
 (175, 2, 212, 192, 110, 213, 105, 112, 83, 168,  
 69, 30, 118, 73, 177, 18, 126, 15, 37, 48)<sub>S5</sub>,  
 (177, 206, 185, 10, 218, 176, 71, 174, 94, 95,  
 151, 21, 126, 211, 104, 111, 169, 67, 209, 166)<sub>S5</sub>,  
 (121, 72, 212, 0, 137, 199, 217, 58, 106, 7,  
 19, 172, 181, 52, 129, 108, 104, 73, 156, 92)<sub>S5</sub>

and

$(\infty, 98, 56, 51, 152, 203, 25, 186, 102, 201,$   
 $154, 22, 189, 143, 61, 141, 155, 13, 1, 147)_{S_6},$   
 $(143, 182, 7, 216, 84, 140, 112, 207, 164, 201,$   
 $53, 81, 205, 122, 115, 82, 200, 51, 1, 173)_{S_6},$   
 $(17, 212, 124, 137, 55, 167, 8, 25, 141, 166,$   
 $11, 164, 108, 139, 91, 105, 37, 46, 112, 38)_{S_6},$   
 $(49, 89, 207, 36, 47, 174, 27, 35, 6, 26,$   
 $67, 211, 46, 76, 19, 215, 122, 87, 199, 132)_{S_6},$   
 $(28, 74, 96, 212, 110, 78, 89, 62, 140, 43,$   
 $156, 95, 64, 166, 106, 162, 190, 67, 198, 49)_{S_6},$   
 $(209, 39, 92, 74, 47, 150, 84, 44, 130, 190,$   
 $79, 73, 137, 3, 180, 140, 204, 23, 14, 29)_{S_6},$   
 $(108, 204, 20, 62, 75, 136, 91, 0, 124, 112,$   
 $175, 60, 96, 149, 187, 202, 170, 5, 101, 177)_{S_6},$   
 $(140, 57, 84, 205, 49, 196, 21, 207, 186, 52,$   
 $69, 75, 81, 198, 164, 178, 62, 79, 115, 152)_{S_6},$   
 $(170, 32, 194, 201, 146, 147, 46, 142, 91, 99,$   
 $72, 206, 116, 202, 172, 95, 84, 112, 18, 75)_{S_6},$   
 $(163, 151, 85, 18, 17, 53, 27, 150, 36, 111,$   
 $204, 145, 174, 24, 205, 141, 83, 176, 164, 16)_{S_6},$   
 $(31, 99, 180, 137, 143, 3, 214, 145, 118, 211,$   
 $111, 5, 120, 57, 37, 107, 142, 218, 45, 200)_{S_6}$

under the action of the mapping  $\infty \mapsto \infty, x \mapsto x + 3 \pmod{219}$ .  $\square$

**Lemma 4.2** *There exist decompositions of the complete multipartite graphs  $K_{5,5,5,5}, K_{10,10,10,10}, K_{6,6,6,6,6}, K_{60,60,75}, K_{60,60,60,75}, K_{15,15,15,21}, K_{24,24,24,24,39}$  and  $K_{24,24,24,24,24,60}$  into each of the six snarks on 20 vertices.*

**Proof.** Let the vertex set of  $K_{5,5,5,5}$  be  $Z_{20}$  partitioned according to residue classes modulo 4. The decomposition consists of

$(14, 3, 17, 9, 15, 13, 18, 0, 8, 2, 1, 5, 11, 6, 16, 19, 12, 10, 4, 7)_{S_1},$   
 $(3, 4, 16, 5, 13, 7, 2, 8, 0, 6, 17, 11, 18, 12, 1, 19, 14, 10, 15, 9)_{S_2},$   
 $(2, 15, 1, 9, 3, 6, 0, 8, 18, 4, 7, 17, 12, 5, 10, 13, 19, 11, 14, 16)_{S_3},$   
 $(12, 1, 14, 6, 15, 8, 3, 11, 13, 4, 7, 10, 18, 2, 5, 0, 19, 9, 17, 16)_{S_4},$   
 $(12, 9, 2, 18, 0, 3, 1, 8, 13, 4, 17, 15, 16, 14, 5, 6, 19, 7, 11, 10)_{S_5}$

and

$(14, 7, 1, 13, 8, 6, 17, 2, 11, 16, 4, 3, 10, 15, 18, 5, 12, 0, 19, 9)_{S_6}$

under the action of the mapping  $x \mapsto x + 4 \pmod{20}$ .

Let the vertex set of  $K_{10,10,10,10}$  be  $Z_{40}$  partitioned according to residue classes modulo 4. The decomposition consists of

$(16, 35, 30, 32, 26, 37, 14, 7, 11, 17, 18, 15, 29, 20, 34, 33, 3, 36, 38, 27)_{S1}$ ,  
 $(13, 0, 11, 33, 7, 8, 30, 36, 17, 10, 5, 26, 1, 3, 25, 15, 16, 12, 20, 22)_{S2}$ ,  
 $(12, 35, 5, 26, 25, 4, 17, 24, 11, 3, 33, 34, 6, 27, 32, 7, 14, 9, 10, 16)_{S3}$ ,  
 $(4, 18, 14, 23, 36, 33, 7, 29, 27, 3, 26, 28, 8, 11, 12, 31, 5, 2, 1, 24)_{S4}$ ,  
 $(29, 11, 25, 23, 30, 8, 6, 4, 21, 26, 22, 31, 28, 3, 14, 7, 27, 18, 17, 12)_{S5}$   
 and  
 $(20, 6, 13, 30, 28, 29, 35, 1, 11, 14, 16, 3, 8, 34, 36, 5, 7, 39, 2, 21)_{S6}$   
 under the action of the mapping  $x \mapsto x + 2 \pmod{40}$ .

Let the vertex set of  $K_{6,6,6,6}$  be  $Z_{30}$  partitioned according to residue classes modulo 5. The decomposition consists of

$(7, 26, 22, 28, 6, 15, 21, 3, 24, 2, 25, 5, 11, 8, 17, 4, 23, 9, 27, 1)_{S1}$ ,  
 $(5, 1, 20, 4, 9, 15, 28, 13, 25, 26, 24, 19, 12, 29, 0, 27, 8, 16, 3, 22)_{S1}$ ,  
 $(23, 20, 25, 4, 14, 12, 28, 2, 16, 3, 22, 18, 27, 0, 21, 10, 9, 19, 1, 8)_{S2}$ ,  
 $(28, 1, 15, 12, 24, 5, 11, 18, 4, 7, 17, 26, 29, 21, 13, 25, 20, 14, 22, 16)_{S2}$ ,  
 $(24, 5, 21, 12, 3, 14, 7, 17, 1, 18, 11, 20, 16, 10, 4, 8, 25, 23, 15, 2)_{S3}$ ,  
 $(25, 3, 2, 28, 23, 11, 9, 15, 14, 4, 12, 21, 24, 8, 22, 20, 26, 27, 1, 19)_{S3}$ ,  
 $(21, 18, 0, 19, 17, 5, 13, 6, 4, 16, 24, 10, 23, 8, 20, 7, 12, 27, 26, 9)_{S4}$ ,  
 $(25, 28, 4, 0, 2, 8, 19, 24, 1, 12, 14, 6, 26, 23, 10, 17, 11, 22, 20, 13)_{S4}$ ,  
 $(4, 23, 1, 16, 10, 9, 6, 22, 3, 27, 26, 8, 20, 15, 21, 28, 25, 18, 7, 19)_{S5}$ ,  
 $(0, 22, 16, 7, 15, 2, 4, 1, 24, 13, 20, 17, 14, 9, 25, 8, 11, 23, 12, 19)_{S5}$   
 and

$(11, 2, 19, 17, 26, 27, 18, 14, 15, 23, 13, 12, 1, 24, 16, 0, 10, 3, 4, 6)_{S6}$ ,  
 $(25, 18, 20, 28, 3, 15, 12, 1, 4, 14, 23, 17, 26, 11, 24, 5, 2, 22, 19, 10)_{S6}$   
 under the action of the mapping  $x \mapsto x + 5 \pmod{30}$ .

Let the vertex set of  $K_{60,60,75}$  be  $\{0, 1, \dots, 194\}$  partitioned into  $\{3j : j = 0, 1, \dots, 59\}$ ,  $\{3j + 1 : j = 0, 1, \dots, 59\}$  and  $\{3j + 2 : j = 0, 1, \dots, 59\} \cup \{180, 181, \dots, 194\}$ . The decomposition consists of

$(81, 8, 27, 0, 186, 151, 86, 107, 169, 89,$   
 $124, 37, 9, 173, 6, 105, 13, 68, 166, 115)_{S1}$ ,  
 $(9, 190, 169, 66, 113, 78, 76, 53, 127, 65,$   
 $114, 21, 84, 186, 163, 121, 143, 19, 131, 135)_{S1}$ ,  
 $(127, 27, 85, 121, 156, 17, 18, 15, 106, 183,$   
 $55, 124, 47, 153, 149, 71, 138, 130, 174, 48)_{S1}$ ,  
 $(55, 62, 1, 115, 95, 124, 193, 92, 25, 108,$   
 $91, 157, 104, 144, 137, 189, 9, 14, 174, 165)_{S1}$ ,  
 $(42, 65, 39, 54, 118, 152, 139, 130, 182, 174,$   
 $107, 47, 102, 189, 19, 61, 80, 66, 11, 56)_{S1}$ ,  
 $(39, 131, 172, 157, 106, 125, 42, 177, 95, 64,$   
 $107, 158, 19, 92, 100, 36, 161, 163, 86, 167)_{S1}$ ,



(146, 0, 149, 47, 3, 180, 15, 87, 148, 30,  
 31, 187, 75, 88, 66, 63, 49, 90, 142, 43)<sub>S1</sub>,  
 (61, 153, 134, 4, 191, 21, 13, 18, 159, 82,  
 148, 189, 81, 121, 39, 26, 92, 129, 143, 88)<sub>S2</sub>,  
 (154, 146, 186, 175, 93, 152, 94, 137, 159, 73,  
 28, 187, 99, 12, 121, 47, 26, 144, 119, 124)<sub>S2</sub>,  
 (147, 10, 17, 69, 137, 13, 39, 34, 98, 117,  
 72, 4, 183, 140, 162, 178, 40, 168, 1, 80)<sub>S2</sub>,  
 (144, 191, 53, 91, 10, 92, 135, 26, 29, 18,  
 172, 96, 192, 55, 165, 142, 128, 6, 15, 28)<sub>S2</sub>,  
 (113, 159, 139, 46, 3, 84, 107, 35, 61, 9,  
 161, 42, 13, 191, 135, 104, 133, 111, 168, 83)<sub>S2</sub>,  
 (14, 177, 106, 188, 129, 88, 180, 136, 162, 23,  
 82, 50, 121, 113, 64, 140, 125, 102, 42, 91)<sub>S2</sub>,  
 (14, 30, 7, 176, 108, 31, 164, 55, 52, 38,  
 146, 15, 11, 105, 26, 12, 136, 167, 178, 45)<sub>S2</sub>,  
 (71, 93, 29, 25, 11, 154, 138, 56, 21, 108,  
 168, 55, 15, 64, 18, 151, 67, 183, 137, 177)<sub>S3</sub>,  
 (48, 40, 191, 92, 119, 114, 173, 63, 136, 10,  
 154, 104, 126, 167, 28, 137, 60, 124, 117, 192)<sub>S3</sub>,  
 (18, 178, 89, 26, 27, 61, 41, 104, 79, 148,  
 68, 120, 149, 168, 77, 154, 3, 139, 39, 180)<sub>S3</sub>,  
 (32, 157, 96, 108, 117, 47, 94, 192, 139, 35,  
 115, 78, 86, 92, 64, 9, 125, 111, 191, 73)<sub>S3</sub>,  
 (187, 156, 67, 55, 173, 94, 165, 128, 90, 66,  
 83, 79, 159, 61, 5, 154, 93, 146, 50, 31)<sub>S3</sub>,  
 (88, 81, 13, 129, 26, 61, 78, 128, 28, 187,  
 36, 94, 111, 139, 56, 16, 174, 29, 32, 58)<sub>S3</sub>,  
 (0, 17, 25, 190, 113, 162, 106, 147, 49, 105,  
 128, 55, 81, 160, 66, 134, 92, 188, 63, 4)<sub>S3</sub>,  
 (38, 84, 64, 166, 2, 55, 138, 131, 120, 93,  
 183, 94, 14, 16, 92, 163, 150, 156, 188, 61)<sub>S4</sub>,  
 (158, 150, 160, 44, 45, 193, 141, 22, 143, 129,  
 66, 92, 149, 14, 54, 82, 73, 115, 50, 105)<sub>S4</sub>,  
 (189, 7, 90, 71, 178, 20, 36, 81, 47, 65,  
 141, 154, 135, 175, 95, 91, 10, 131, 92, 78)<sub>S4</sub>,  
 (95, 105, 43, 8, 49, 63, 67, 148, 168, 32,  
 70, 45, 39, 65, 7, 20, 57, 22, 171, 62)<sub>S4</sub>,  
 (12, 91, 169, 110, 160, 104, 49, 45, 182, 71,  
 0, 109, 148, 116, 114, 181, 22, 21, 171, 140)<sub>S4</sub>,

(50, 144, 118, 77, 124, 104, 58, 14, 129, 185,  
 69, 4, 13, 101, 24, 189, 163, 60, 117, 184)<sub>S4</sub>,  
 (0, 22, 49, 8, 120, 7, 152, 57, 164, 135,  
 9, 127, 183, 101, 4, 84, 30, 76, 46, 53)<sub>S4</sub>,  
 (36, 52, 168, 43, 155, 103, 77, 98, 60, 65,  
 48, 70, 93, 20, 31, 32, 51, 85, 21, 41)<sub>S5</sub>,  
 (133, 164, 61, 167, 29, 120, 119, 39, 79, 105,  
 122, 124, 189, 142, 62, 49, 55, 69, 3, 95)<sub>S5</sub>,  
 (164, 18, 22, 171, 27, 155, 160, 124, 183, 190,  
 136, 54, 121, 113, 7, 159, 66, 108, 122, 4)<sub>S5</sub>,  
 (185, 118, 110, 30, 100, 167, 36, 69, 65, 157,  
 120, 37, 62, 49, 126, 25, 59, 138, 95, 163)<sub>S5</sub>,  
 (95, 63, 22, 141, 36, 70, 126, 61, 158, 181,  
 117, 5, 135, 175, 104, 145, 177, 180, 86, 28)<sub>S5</sub>,  
 (81, 161, 132, 133, 97, 123, 188, 177, 183, 38,  
 176, 112, 186, 15, 125, 42, 117, 4, 79, 95)<sub>S5</sub>,  
 (0, 1, 87, 88, 148, 95, 112, 162, 194, 131,  
 46, 119, 21, 125, 31, 86, 60, 64, 12, 35)<sub>S5</sub>

and

(67, 102, 152, 161, 115, 44, 75, 187, 114, 85,  
 76, 51, 96, 137, 21, 173, 4, 157, 37, 33)<sub>S6</sub>,  
 (187, 174, 112, 90, 6, 184, 22, 14, 70, 185,  
 4, 171, 13, 186, 166, 138, 170, 150, 11, 118)<sub>S6</sub>,  
 (65, 130, 116, 84, 73, 99, 115, 30, 143, 42,  
 148, 44, 185, 33, 20, 105, 178, 55, 14, 132)<sub>S6</sub>,  
 (57, 122, 18, 7, 47, 117, 83, 160, 129, 181,  
 65, 81, 99, 142, 66, 98, 41, 140, 91, 84)<sub>S6</sub>,  
 (91, 105, 46, 11, 77, 124, 173, 149, 103, 143,  
 83, 132, 3, 12, 148, 193, 57, 85, 157, 140)<sub>S6</sub>,  
 (41, 96, 104, 28, 109, 181, 115, 73, 156, 126,  
 97, 38, 13, 189, 16, 18, 69, 174, 125, 106)<sub>S6</sub>,  
 (0, 5, 4, 119, 32, 148, 86, 84, 172, 95,  
 39, 124, 163, 161, 136, 41, 126, 93, 51, 89)<sub>S6</sub>

under the action of the mapping  $x \mapsto x + 3 \pmod{180}$  for  $x < 180$ ,  $x \mapsto 180 + (x + 1 \pmod{15})$  for  $x \geq 180$ .

Let the vertex set of  $K_{60,60,60,75}$  be  $\{0, 1, \dots, 254\}$  partitioned into  $\{3j + i : j = 0, 1, \dots, 59\}$ ,  $i = 0, 1, 2$ , and  $\{180, 181, \dots, 254\}$ . The decomposition consists of

(251, 25, 138, 197, 92, 73, 171, 23, 0, 244,  
 157, 58, 44, 232, 122, 29, 21, 164, 109, 127)<sub>S1</sub>,  
 (46, 153, 76, 236, 227, 32, 42, 135, 78, 97,  
 161, 98, 247, 175, 213, 230, 30, 180, 4, 149)<sub>S1</sub>,

(167, 21, 110, 104, 230, 171, 157, 76, 66, 31,  
 36, 249, 253, 176, 168, 233, 160, 214, 165, 172)<sub>S1</sub>,  
 (247, 29, 238, 27, 177, 225, 131, 114, 186, 93,  
 146, 151, 166, 179, 15, 21, 107, 84, 222, 85)<sub>S1</sub>,  
 (187, 175, 5, 42, 82, 199, 1, 96, 209, 63,  
 76, 238, 101, 240, 79, 74, 69, 143, 190, 221)<sub>S1</sub>,  
 (45, 227, 178, 114, 101, 16, 128, 68, 186, 137,  
 184, 229, 121, 3, 140, 97, 53, 183, 125, 171)<sub>S1</sub>,  
 (118, 206, 31, 158, 147, 204, 64, 119, 214, 81,  
 192, 201, 36, 61, 78, 140, 251, 124, 233, 220)<sub>S1</sub>,  
 (3, 145, 5, 36, 104, 220, 83, 70, 46, 117,  
 215, 203, 0, 119, 162, 115, 28, 177, 248, 136)<sub>S1</sub>,  
 (47, 215, 146, 6, 211, 28, 191, 66, 179, 48,  
 83, 26, 189, 50, 242, 0, 76, 238, 104, 72)<sub>S1</sub>,  
 (100, 96, 187, 53, 251, 145, 87, 174, 59, 46,  
 55, 229, 159, 240, 28, 228, 119, 143, 107, 201)<sub>S2</sub>,  
 (142, 174, 171, 73, 181, 197, 175, 113, 125, 207,  
 161, 214, 30, 8, 168, 248, 61, 62, 58, 99)<sub>S2</sub>,  
 (136, 227, 75, 143, 223, 237, 70, 138, 108, 169,  
 71, 97, 246, 67, 59, 144, 115, 134, 135, 200)<sub>S2</sub>,  
 (68, 21, 233, 50, 209, 226, 126, 97, 224, 51,  
 103, 119, 185, 15, 130, 137, 32, 118, 5, 54)<sub>S2</sub>,  
 (116, 132, 192, 201, 36, 13, 80, 82, 194, 68,  
 40, 6, 41, 35, 213, 27, 89, 154, 18, 233)<sub>S2</sub>,  
 (234, 2, 59, 10, 125, 150, 82, 132, 184, 79,  
 163, 96, 235, 113, 233, 167, 45, 135, 24, 110)<sub>S2</sub>,  
 (15, 112, 185, 33, 219, 158, 159, 31, 89, 114,  
 211, 163, 217, 169, 135, 10, 139, 134, 84, 250)<sub>S2</sub>,  
 (241, 84, 120, 158, 127, 157, 162, 228, 247, 89,  
 181, 68, 171, 252, 108, 229, 46, 146, 94, 220)<sub>S2</sub>,  
 (95, 186, 112, 52, 202, 248, 85, 122, 108, 204,  
 19, 198, 13, 12, 10, 86, 230, 250, 190, 9)<sub>S2</sub>,  
 (18, 220, 162, 134, 39, 219, 133, 31, 253, 127,  
 6, 149, 189, 229, 79, 150, 239, 14, 234, 63)<sub>S3</sub>,  
 (216, 16, 114, 31, 185, 152, 93, 235, 27, 248,  
 174, 65, 155, 59, 42, 208, 28, 251, 173, 84)<sub>S3</sub>,  
 (155, 156, 56, 253, 190, 100, 223, 106, 165, 148,  
 204, 114, 136, 107, 40, 47, 9, 69, 37, 176)<sub>S3</sub>,  
 (234, 173, 182, 14, 164, 241, 99, 172, 45, 166,  
 114, 242, 184, 29, 10, 6, 54, 0, 91, 232)<sub>S3</sub>,  
 (124, 27, 250, 23, 3, 85, 149, 16, 202, 79,  
 186, 53, 208, 74, 178, 159, 95, 84, 246, 100)<sub>S3</sub>,

(230, 129, 127, 64, 71, 111, 97, 157, 202, 41,  
 224, 54, 25, 61, 9, 59, 227, 246, 117, 5)<sub>S3</sub>,  
 (187, 112, 164, 45, 203, 48, 8, 207, 178, 94,  
 248, 41, 86, 68, 235, 40, 121, 242, 188, 153)<sub>S3</sub>,  
 (198, 70, 12, 63, 232, 147, 92, 177, 107, 74,  
 43, 221, 55, 68, 180, 20, 118, 194, 250, 141)<sub>S3</sub>,  
 (10, 149, 144, 78, 66, 5, 111, 165, 40, 216,  
 206, 2, 109, 64, 62, 235, 221, 32, 23, 52)<sub>S3</sub>,  
 (165, 253, 50, 99, 58, 9, 17, 57, 248, 226,  
 207, 70, 44, 122, 108, 202, 8, 200, 100, 162)<sub>S4</sub>,  
 (116, 144, 7, 154, 189, 84, 211, 234, 69, 177,  
 118, 161, 11, 176, 124, 210, 101, 40, 57, 205)<sub>S4</sub>,  
 (75, 198, 134, 99, 104, 210, 10, 156, 79, 154,  
 113, 221, 220, 188, 96, 143, 204, 97, 60, 20)<sub>S4</sub>,  
 (170, 64, 243, 244, 108, 145, 189, 66, 164, 19,  
 239, 47, 17, 22, 192, 141, 29, 208, 46, 24)<sub>S4</sub>,  
 (32, 232, 31, 61, 211, 83, 118, 129, 65, 132,  
 54, 248, 247, 223, 124, 3, 131, 99, 191, 157)<sub>S4</sub>,  
 (238, 88, 97, 107, 0, 197, 44, 113, 244, 141,  
 40, 232, 47, 32, 251, 94, 55, 121, 229, 98)<sub>S4</sub>,  
 (40, 186, 150, 63, 17, 204, 53, 87, 34, 215,  
 49, 137, 143, 183, 147, 190, 130, 70, 18, 119)<sub>S4</sub>,  
 (211, 111, 52, 222, 121, 190, 162, 11, 219, 140,  
 104, 171, 102, 9, 149, 220, 0, 113, 28, 187)<sub>S4</sub>,  
 (28, 140, 44, 36, 231, 55, 167, 103, 48, 76,  
 180, 5, 77, 20, 13, 246, 155, 81, 9, 212)<sub>S4</sub>,  
 (166, 182, 20, 60, 148, 193, 81, 155, 234, 181,  
 52, 200, 118, 30, 252, 132, 97, 183, 223, 165)<sub>S5</sub>,  
 (146, 55, 18, 177, 67, 237, 78, 171, 80, 196,  
 136, 89, 46, 219, 94, 53, 63, 122, 235, 166)<sub>S5</sub>,  
 (161, 163, 247, 30, 26, 168, 151, 195, 109, 77,  
 82, 42, 201, 244, 73, 131, 227, 81, 208, 22)<sub>S5</sub>,  
 (177, 16, 0, 166, 201, 18, 106, 83, 123, 146,  
 44, 63, 19, 237, 92, 213, 67, 143, 78, 215)<sub>S5</sub>,  
 (94, 164, 219, 21, 113, 117, 152, 247, 4, 55,  
 178, 123, 217, 2, 235, 85, 77, 67, 72, 253)<sub>S5</sub>,  
 (233, 13, 14, 113, 22, 17, 91, 147, 201, 196,  
 76, 236, 83, 65, 180, 120, 112, 60, 204, 110)<sub>S5</sub>,  
 (212, 85, 216, 75, 92, 224, 160, 155, 196, 140,  
 167, 238, 150, 229, 113, 52, 63, 2, 170, 190)<sub>S5</sub>,  
 (96, 43, 137, 40, 24, 1, 209, 95, 203, 33,  
 192, 10, 235, 76, 243, 4, 178, 101, 5, 230)<sub>S5</sub>,

(13, 167, 126, 101, 155, 229, 21, 94, 254, 1,  
112, 48, 221, 165, 151, 9, 119, 14, 173, 210)<sub>S5</sub>

and

(174, 137, 58, 181, 0, 1, 182, 117, 46, 150,  
50, 22, 207, 61, 245, 152, 34, 38, 126, 221)<sub>S6</sub>,  
(174, 32, 31, 151, 156, 193, 145, 171, 248, 99,  
83, 246, 194, 195, 139, 177, 211, 60, 190, 104)<sub>S6</sub>,  
(79, 114, 4, 191, 12, 187, 14, 183, 37, 24,  
80, 121, 46, 222, 32, 51, 72, 159, 175, 239)<sub>S6</sub>,  
(61, 143, 81, 218, 208, 62, 177, 192, 103, 160,  
138, 47, 174, 184, 135, 149, 178, 97, 145, 216)<sub>S6</sub>,  
(225, 169, 187, 175, 129, 206, 96, 59, 215, 43,  
38, 202, 229, 94, 253, 84, 6, 179, 178, 223)<sub>S6</sub>,  
(201, 29, 90, 134, 124, 141, 163, 82, 117, 127,  
253, 2, 229, 176, 250, 148, 60, 27, 135, 183)<sub>S6</sub>,  
(218, 92, 210, 30, 141, 37, 143, 160, 225, 91,  
196, 96, 62, 166, 116, 209, 163, 229, 130, 120)<sub>S6</sub>,  
(250, 87, 34, 54, 117, 148, 164, 167, 58, 202,  
89, 37, 5, 231, 62, 94, 184, 7, 180, 147)<sub>S6</sub>,  
(6, 55, 249, 67, 177, 131, 124, 45, 212, 9,  
165, 1, 229, 43, 129, 22, 41, 146, 232, 148)<sub>S6</sub>

under the action of the mapping  $x \mapsto x + 2 \pmod{180}$  for  $x < 180$ ,  $x \mapsto 180 + (x - 180 + 5 \pmod{75})$  for  $x \geq 180$ .

Let the vertex set of  $K_{15,15,15,21}$  be  $\{0, 1, \dots, 65\}$  partitioned into  $\{3j + i : j = 0, 1, \dots, 14\}$ ,  $i = 0, 1, 2$ , and  $\{45, 46, \dots, 65\}$ . The decomposition consists of

(57, 0, 48, 51, 18, 11, 31, 6, 28, 3, 47, 50, 55, 36, 7, 40, 1, 64, 24, 14)<sub>S1</sub>,  
(56, 37, 11, 52, 0, 61, 21, 33, 60, 40, 29, 59, 35, 51, 7, 19, 34, 39, 41, 49)<sub>S1</sub>,  
(50, 12, 20, 23, 2, 4, 36, 21, 60, 41, 51, 56, 39, 22, 17, 38, 8, 6, 37, 7)<sub>S1</sub>,  
(21, 46, 9, 3, 13, 29, 60, 52, 5, 62, 4, 23, 45, 38, 0, 12, 10, 17, 48, 47)<sub>S1</sub>,  
(0, 62, 44, 2, 28, 33, 58, 51, 11, 19, 32, 15, 21, 5, 40, 36, 14, 3, 54, 63)<sub>S1</sub>,  
(37, 47, 14, 19, 35, 57, 44, 40, 12, 52, 38, 18, 1, 20, 49, 25, 53, 17, 42, 13)<sub>S1</sub>,  
(14, 16, 47, 45, 0, 38, 63, 44, 7, 9, 20, 3, 4, 15, 58, 32, 28, 31, 56, 24)<sub>S2</sub>,  
(54, 24, 26, 8, 28, 64, 15, 36, 7, 48, 1, 51, 5, 2, 33, 22, 17, 58, 60, 4)<sub>S2</sub>,  
(61, 43, 8, 51, 27, 3, 2, 4, 46, 18, 57, 5, 15, 53, 9, 26, 17, 28, 13, 6)<sub>S2</sub>,  
(37, 47, 60, 20, 56, 39, 5, 42, 62, 40, 12, 38, 19, 48, 27, 45, 17, 6, 8, 59)<sub>S2</sub>,  
(17, 45, 24, 39, 30, 50, 10, 62, 15, 44, 25, 59, 28, 16, 48, 33, 32, 49, 64, 36)<sub>S2</sub>,  
(0, 4, 1, 65, 47, 6, 11, 40, 63, 33, 44, 36, 10, 31, 12, 26, 64, 60, 41, 9)<sub>S2</sub>,  
(21, 35, 27, 55, 57, 22, 54, 63, 30, 37, 8, 6, 43, 61, 34, 5, 23, 39, 15, 62)<sub>S3</sub>,  
(47, 20, 51, 21, 0, 17, 12, 40, 60, 41, 61, 6, 43, 33, 53, 38, 36, 45, 25, 44)<sub>S3</sub>,  
(34, 39, 8, 35, 57, 27, 48, 46, 4, 18, 55, 38, 17, 13, 41, 63, 64, 49, 9, 19)<sub>S3</sub>,  
(35, 55, 31, 39, 60, 9, 45, 26, 7, 23, 20, 57, 16, 58, 38, 27, 51, 12, 4, 29)<sub>S3</sub>,

$(54, 39, 57, 38, 29, 27, 5, 26, 31, 16, 51, 9, 22, 19, 41, 59, 25, 63, 10, 33)_{S_3}$ ,  
 $(14, 33, 1, 13, 10, 6, 22, 32, 51, 63, 45, 25, 36, 37, 18, 59, 2, 35, 42, 60)_{S_3}$ ,  
 $(28, 47, 42, 6, 44, 19, 45, 30, 40, 10, 36, 23, 5, 17, 16, 64, 62, 25, 24, 32)_{S_4}$ ,  
 $(57, 5, 8, 19, 20, 45, 21, 52, 1, 43, 10, 27, 61, 49, 33, 32, 60, 24, 53, 40)_{S_4}$ ,  
 $(6, 2, 43, 22, 64, 17, 55, 34, 44, 59, 27, 11, 9, 61, 35, 51, 56, 37, 30, 26)_{S_4}$ ,  
 $(34, 8, 36, 64, 0, 23, 18, 4, 58, 63, 45, 2, 11, 12, 62, 40, 3, 51, 29, 22)_{S_4}$ ,  
 $(32, 28, 46, 62, 4, 64, 27, 11, 36, 29, 6, 61, 60, 16, 53, 38, 30, 2, 49, 13)_{S_4}$ ,  
 $(43, 23, 41, 9, 35, 12, 65, 15, 1, 18, 34, 61, 58, 48, 44, 39, 63, 28, 31, 30)_{S_4}$ ,  
 $(51, 17, 7, 40, 20, 15, 41, 64, 9, 61, 1, 33, 60, 62, 21, 19, 11, 13, 52, 24)_{S_5}$ ,  
 $(61, 17, 28, 18, 58, 8, 24, 4, 20, 56, 36, 63, 41, 47, 43, 64, 42, 25, 7, 45)_{S_5}$ ,  
 $(31, 9, 14, 63, 46, 40, 36, 5, 30, 28, 18, 43, 39, 41, 59, 13, 33, 53, 11, 7)_{S_5}$ ,  
 $(62, 12, 35, 37, 43, 50, 15, 28, 36, 51, 19, 48, 29, 0, 59, 27, 21, 7, 32, 60)_{S_5}$ ,  
 $(46, 35, 52, 43, 44, 54, 1, 0, 61, 33, 7, 56, 19, 15, 14, 58, 34, 49, 17, 42)_{S_5}$ ,  
 $(2, 33, 5, 36, 16, 48, 0, 37, 44, 64, 25, 57, 29, 26, 12, 55, 31, 19, 13, 51)_{S_5}$

and

$(6, 2, 13, 14, 48, 21, 35, 59, 16, 17, 63, 12, 30, 5, 28, 44, 64, 47, 56, 25)_{S_6}$ ,  
 $(54, 33, 40, 23, 58, 3, 11, 32, 1, 14, 43, 9, 25, 49, 21, 57, 53, 27, 18, 16)_{S_6}$ ,  
 $(59, 8, 4, 36, 53, 15, 13, 60, 40, 55, 41, 12, 43, 1, 39, 35, 26, 18, 42, 48)_{S_6}$ ,  
 $(36, 56, 8, 64, 21, 54, 14, 27, 1, 58, 37, 44, 26, 24, 43, 15, 33, 25, 55, 20)_{S_6}$ ,  
 $(63, 33, 61, 29, 21, 4, 39, 34, 52, 44, 20, 42, 47, 58, 43, 12, 53, 22, 64, 32)_{S_6}$ ,  
 $(0, 58, 9, 20, 50, 12, 37, 27, 10, 65, 53, 26, 7, 39, 19, 33, 45, 55, 59, 25)_{S_6}$

under the action of the mapping  $x \mapsto x + 5 \pmod{45}$  for  $x < 45$ ,  $x \mapsto 45 + (x - 45 + 7 \pmod{21})$  for  $x \geq 45$ .

Let the vertex set of  $K_{24,24,24,24,39}$  be  $\{0, 1, \dots, 134\}$  partitioned into  $\{4j + i : j = 0, 1, \dots, 23\}$ ,  $i = 0, 1, 2, 3$ , and  $\{96, 97, \dots, 134\}$ . The decomposition consists of

$(0, 124, 68, 29, 116, 45, 63, 47, 35, 132,$   
 $38, 93, 133, 44, 107, 67, 87, 56, 46, 30)_{S_1}$ ,  
 $(75, 17, 112, 60, 49, 121, 82, 51, 48, 123,$   
 $10, 33, 18, 87, 36, 102, 125, 92, 29, 38)_{S_1}$ ,  
 $(94, 43, 30, 90, 23, 92, 1, 122, 42, 57,$   
 $47, 124, 31, 126, 70, 54, 97, 87, 93, 129)_{S_1}$ ,  
 $(26, 119, 88, 5, 105, 0, 34, 117, 41, 95,$   
 $49, 84, 109, 19, 113, 72, 44, 31, 127, 9)_{S_1}$ ,  
 $(14, 33, 74, 19, 117, 63, 78, 68, 29, 64,$   
 $31, 39, 101, 4, 105, 134, 0, 102, 20, 13)_{S_1}$ ,  
 $(128, 47, 26, 102, 6, 24, 59, 15, 12, 131,$   
 $62, 7, 16, 107, 65, 124, 96, 57, 38, 55)_{S_2}$ ,  
 $(68, 30, 5, 89, 34, 133, 80, 112, 73, 18,$   
 $35, 100, 28, 25, 58, 57, 110, 88, 42, 19)_{S_2}$ ,

(93, 82, 74, 4, 68, 43, 130, 94, 30, 114,  
 119, 75, 54, 108, 21, 106, 55, 64, 24, 121) $_{S_2}$ ,  
 (93, 23, 15, 62, 8, 69, 118, 85, 65, 101,  
 129, 14, 98, 39, 74, 76, 21, 96, 81, 6) $_{S_2}$ ,  
 (75, 130, 100, 83, 1, 33, 91, 99, 5, 30,  
 4, 116, 62, 32, 29, 15, 58, 74, 35, 123) $_{S_2}$ ,  
 (48, 57, 92, 39, 124, 20, 131, 80, 85, 122,  
 121, 29, 34, 87, 44, 75, 94, 114, 69, 68) $_{S_3}$ ,  
 (81, 46, 79, 97, 2, 129, 27, 68, 83, 14,  
 12, 39, 107, 6, 45, 7, 92, 57, 44, 54) $_{S_3}$ ,  
 (89, 79, 4, 34, 10, 3, 113, 26, 47, 33,  
 104, 44, 87, 22, 84, 110, 128, 76, 86, 83) $_{S_3}$ ,  
 (103, 67, 132, 72, 130, 81, 15, 98, 95, 39,  
 86, 20, 70, 118, 59, 107, 2, 45, 42, 96) $_{S_3}$ ,  
 (43, 121, 28, 21, 34, 99, 61, 78, 125, 101,  
 3, 29, 113, 128, 44, 30, 119, 79, 47, 45) $_{S_3}$ ,  
 (77, 7, 115, 97, 32, 69, 23, 60, 70, 6,  
 48, 79, 119, 53, 122, 37, 24, 71, 94, 113) $_{S_4}$ ,  
 (110, 60, 71, 21, 32, 39, 61, 26, 5, 114,  
 105, 12, 13, 85, 40, 120, 15, 76, 38, 103) $_{S_4}$ ,  
 (26, 25, 37, 42, 83, 92, 73, 112, 24, 107,  
 131, 51, 40, 30, 125, 58, 13, 132, 56, 23) $_{S_4}$ ,  
 (126, 40, 71, 93, 117, 7, 45, 10, 16, 84,  
 69, 104, 81, 98, 59, 108, 66, 67, 44, 101) $_{S_4}$ ,  
 (75, 26, 121, 116, 57, 42, 77, 98, 72, 44,  
 20, 70, 86, 15, 82, 51, 37, 122, 128, 34) $_{S_4}$ ,  
 (15, 29, 97, 130, 58, 36, 61, 18, 48, 13,  
 37, 7, 70, 127, 27, 106, 109, 16, 19, 53) $_{S_5}$ ,  
 (51, 32, 59, 53, 41, 42, 132, 47, 0, 36,  
 48, 102, 27, 104, 1, 34, 112, 46, 52, 17) $_{S_5}$ ,  
 (125, 59, 64, 48, 50, 23, 16, 72, 93, 98,  
 103, 7, 45, 0, 94, 118, 32, 105, 42, 71) $_{S_5}$ ,  
 (127, 47, 110, 90, 46, 109, 44, 88, 43, 32,  
 24, 126, 38, 133, 5, 95, 128, 15, 120, 12) $_{S_5}$ ,  
 (74, 53, 63, 83, 72, 55, 121, 40, 116, 111,  
 20, 108, 9, 66, 51, 46, 73, 35, 21, 100) $_{S_5}$   
 and  
 (78, 116, 74, 43, 126, 89, 115, 23, 60, 77,  
 22, 48, 80, 24, 54, 67, 110, 62, 20, 53) $_{S_6}$ ,  
 (9, 118, 31, 95, 107, 59, 108, 50, 84, 54,  
 124, 39, 7, 60, 82, 119, 1, 131, 12, 2) $_{S_6}$ ,

$(25, 46, 115, 64, 21, 39, 4, 55, 14, 91,$   
 $61, 36, 28, 7, 66, 98, 117, 126, 57, 51)_{S_6},$   
 $(23, 103, 13, 9, 125, 8, 99, 80, 35, 70,$   
 $110, 75, 107, 4, 49, 68, 122, 21, 54, 48)_{S_6},$   
 $(19, 96, 0, 69, 43, 101, 18, 40, 130, 31,$   
 $2, 108, 61, 119, 11, 114, 42, 3, 72, 65)_{S_6}$   
 under the action of the mapping  $x \mapsto x + 2 \pmod{96}$  for  $x < 96$ ,  $x \mapsto$   
 $96 + (x - 96 + 13 \pmod{39})$  for  $x \geq 96$ .

Let the vertex set of  $K_{24,24,24,24,24,60}$  be  $\{0, 1, \dots, 203\}$  partitioned  
 into  $\{6j + i : j = 0, 1, \dots, 23\}$ ,  $i = 0, 1, \dots, 5$ , and  $\{144, 145, \dots, 203\}$ . The  
 decomposition consists of

$(130, 126, 170, 37, 198, 66, 131, 68, 29, 169,$   
 $117, 75, 171, 110, 17, 179, 19, 201, 16, 134)_{S_1},$   
 $(62, 184, 85, 103, 22, 75, 23, 104, 109, 155,$   
 $28, 122, 191, 99, 157, 198, 70, 129, 60, 59)_{S_1},$   
 $(30, 50, 198, 132, 74, 33, 142, 47, 27, 12,$   
 $15, 75, 180, 86, 202, 61, 101, 40, 172, 49)_{S_1},$   
 $(134, 166, 119, 15, 168, 133, 128, 194, 51, 23,$   
 $6, 0, 172, 10, 143, 44, 95, 175, 22, 12)_{S_1},$   
 $(131, 10, 14, 41, 135, 114, 185, 106, 95, 191,$   
 $102, 88, 109, 64, 123, 173, 22, 29, 122, 152)_{S_2},$   
 $(157, 33, 34, 59, 19, 193, 14, 199, 123, 121,$   
 $192, 143, 110, 128, 169, 54, 39, 131, 73, 164)_{S_2},$   
 $(90, 135, 124, 175, 176, 131, 120, 142, 25, 125,$   
 $72, 166, 83, 173, 24, 14, 33, 96, 80, 58)_{S_2},$   
 $(169, 57, 131, 17, 22, 78, 181, 63, 101, 190,$   
 $4, 170, 126, 193, 61, 162, 80, 38, 142, 6)_{S_2},$   
 $(168, 82, 154, 2, 159, 6, 43, 36, 89, 33,$   
 $35, 44, 32, 104, 94, 111, 109, 19, 151, 0)_{S_3},$   
 $(122, 16, 127, 163, 75, 180, 70, 164, 41, 111,$   
 $64, 61, 101, 140, 97, 74, 8, 69, 190, 71)_{S_3},$   
 $(92, 157, 63, 148, 179, 59, 186, 40, 137, 91,$   
 $54, 192, 49, 156, 15, 71, 22, 48, 75, 115)_{S_3},$   
 $(80, 201, 4, 124, 37, 95, 165, 115, 187, 103,$   
 $15, 54, 57, 29, 99, 160, 147, 188, 2, 84)_{S_3},$   
 $(13, 98, 32, 55, 6, 155, 122, 145, 34, 103,$   
 $92, 35, 149, 156, 51, 4, 82, 81, 202, 104)_{S_4},$   
 $(101, 39, 51, 77, 36, 23, 98, 130, 181, 161,$   
 $156, 49, 125, 102, 119, 194, 85, 88, 31, 153)_{S_4},$   
 $(22, 66, 18, 174, 89, 94, 54, 86, 190, 8,$   
 $48, 183, 29, 187, 79, 118, 46, 117, 119, 153)_{S_4},$



$(61, 158, 10, 112, 123, 155, 45, 42, 71, 147,$   
 $17, 116, 6, 81, 90, 64, 97, 164, 157, 122)_{S_4},$   
 $(170, 5, 54, 76, 157, 9, 74, 195, 116, 8,$   
 $112, 120, 113, 189, 34, 172, 146, 97, 135, 78)_{S_5},$   
 $(105, 58, 158, 189, 50, 71, 178, 148, 83, 32,$   
 $170, 45, 82, 133, 44, 154, 102, 135, 79, 130)_{S_5},$   
 $(152, 88, 127, 46, 113, 50, 33, 161, 36, 3,$   
 $82, 55, 173, 120, 51, 48, 26, 21, 133, 156)_{S_5},$   
 $(145, 13, 2, 15, 146, 63, 41, 4, 77, 28,$   
 $7, 3, 34, 36, 71, 192, 149, 86, 56, 22)_{S_5}$

and

$(96, 154, 97, 140, 164, 112, 127, 19, 101, 201,$   
 $36, 27, 114, 145, 32, 85, 141, 136, 143, 151)_{S_6},$   
 $(27, 73, 168, 46, 131, 129, 133, 105, 196, 125,$   
 $153, 36, 102, 3, 134, 107, 113, 91, 62, 202)_{S_6},$   
 $(116, 106, 159, 142, 9, 72, 123, 162, 92, 35,$   
 $189, 109, 156, 95, 190, 68, 114, 31, 202, 11)_{S_6},$   
 $(168, 122, 79, 46, 32, 163, 127, 119, 192, 65,$   
 $3, 40, 8, 37, 20, 112, 131, 180, 190, 36)_{S_6}$

under the action of the mapping  $x \mapsto x + 1 \pmod{144}$  for  $x < 144$ ,  $x \mapsto 144 + (x - 144 + 5 \pmod{60})$  for  $x \geq 144$ .  $\square$

Theorem 1.3 follows from Lemmas 4.1, 4.2 and proposition 1.6.

## 5 The twenty snarks on 22 vertices

There are twenty snarks on 22 vertices. We represent them by ordered 22-tuples of vertices:  $(1, 2, \dots, 22)_{L_1}, (1, 2, \dots, 22)_{L_2}, \dots, (1, 2, \dots, 22)_{L_{20}}$ . The edge sets are respectively

$L_1: \{\{1, 2\}, \{1, 3\}, \{1, 10\}, \{2, 4\}, \{2, 21\}, \{3, 6\}, \{3, 7\}, \{4, 5\}, \{4, 7\},$   
 $\{5, 6\}, \{5, 8\}, \{6, 17\}, \{7, 11\}, \{8, 13\}, \{8, 15\}, \{9, 10\}, \{9, 12\}, \{9, 15\},$   
 $\{10, 14\}, \{11, 12\}, \{11, 16\}, \{12, 13\}, \{13, 14\}, \{14, 22\}, \{15, 18\}, \{16, 19\},$   
 $\{16, 20\}, \{17, 18\}, \{17, 20\}, \{18, 19\}, \{19, 21\}, \{20, 22\}, \{21, 22\}\},$

$L_2: \{\{1, 2\}, \{1, 3\}, \{1, 10\}, \{2, 4\}, \{2, 17\}, \{3, 6\}, \{3, 7\}, \{4, 5\}, \{4, 7\},$   
 $\{5, 6\}, \{5, 8\}, \{6, 21\}, \{7, 11\}, \{8, 13\}, \{8, 15\}, \{9, 10\}, \{9, 12\}, \{9, 15\},$   
 $\{10, 14\}, \{11, 12\}, \{11, 16\}, \{12, 13\}, \{13, 14\}, \{14, 22\}, \{15, 18\}, \{16, 19\},$   
 $\{16, 20\}, \{17, 18\}, \{17, 20\}, \{18, 19\}, \{19, 21\}, \{20, 22\}, \{21, 22\}\},$

$L_3: \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{3, 7\}, \{3, 8\}, \{4, 9\}, \{4, 10\},$   
 $\{5, 7\}, \{5, 9\}, \{6, 8\}, \{6, 10\}, \{7, 11\}, \{8, 12\}, \{9, 13\}, \{10, 14\}, \{11, 14\},$   
 $\{11, 15\}, \{12, 13\}, \{12, 16\}, \{13, 17\}, \{14, 18\}, \{15, 19\}, \{15, 20\}, \{16, 19\},$   
 $\{16, 21\}, \{17, 20\}, \{17, 22\}, \{18, 21\}, \{18, 22\}, \{19, 22\}, \{20, 21\}\},$



$\{16, 19\}, \{16, 20\}, \{17, 21\}, \{17, 22\}, \{18, 22\}, \{19, 21\}, \{20, 22\}\},$   
 L15:  $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{3, 7\}, \{3, 8\}, \{4, 9\}, \{4, 10\},$   
 $\{5, 7\}, \{5, 9\}, \{6, 11\}, \{6, 12\}, \{7, 10\}, \{8, 11\}, \{8, 13\}, \{9, 14\}, \{10, 15\},$   
 $\{11, 16\}, \{12, 15\}, \{12, 17\}, \{13, 14\}, \{13, 18\}, \{14, 19\}, \{15, 20\}, \{16, 21\},$   
 $\{16, 22\}, \{17, 18\}, \{17, 21\}, \{18, 22\}, \{19, 20\}, \{19, 21\}, \{20, 22\}\},$   
 L16:  $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{3, 7\}, \{3, 8\}, \{4, 9\}, \{4, 10\},$   
 $\{5, 7\}, \{5, 9\}, \{6, 11\}, \{6, 12\}, \{7, 10\}, \{8, 11\}, \{8, 13\}, \{9, 14\}, \{10, 15\},$   
 $\{11, 16\}, \{12, 15\}, \{12, 17\}, \{13, 14\}, \{13, 18\}, \{14, 19\}, \{15, 20\}, \{16, 21\},$   
 $\{16, 22\}, \{17, 19\}, \{17, 21\}, \{18, 20\}, \{18, 21\}, \{19, 22\}, \{20, 22\}\},$   
 L17:  $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{3, 7\}, \{3, 8\}, \{4, 9\}, \{4, 10\},$   
 $\{5, 7\}, \{5, 9\}, \{6, 11\}, \{6, 12\}, \{7, 10\}, \{8, 11\}, \{8, 13\}, \{9, 14\}, \{10, 15\},$   
 $\{11, 16\}, \{12, 15\}, \{12, 17\}, \{13, 18\}, \{13, 19\}, \{14, 20\}, \{14, 21\}, \{15, 16\},$   
 $\{16, 22\}, \{17, 18\}, \{17, 20\}, \{18, 21\}, \{19, 20\}, \{19, 22\}, \{21, 22\}\},$   
 L18:  $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{3, 7\}, \{3, 8\}, \{4, 9\}, \{4, 10\},$   
 $\{5, 7\}, \{5, 9\}, \{6, 11\}, \{6, 12\}, \{7, 10\}, \{8, 13\}, \{8, 14\}, \{9, 15\}, \{10, 16\},$   
 $\{11, 13\}, \{11, 16\}, \{12, 17\}, \{12, 18\}, \{13, 15\}, \{14, 17\}, \{14, 19\}, \{15, 20\},$   
 $\{16, 21\}, \{17, 22\}, \{18, 19\}, \{18, 20\}, \{19, 21\}, \{20, 22\}, \{21, 22\}\},$   
 L19:  $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{3, 7\}, \{3, 8\}, \{4, 9\}, \{4, 10\},$   
 $\{5, 7\}, \{5, 9\}, \{6, 11\}, \{6, 12\}, \{7, 10\}, \{8, 13\}, \{8, 14\}, \{9, 15\}, \{10, 16\},$   
 $\{11, 13\}, \{11, 16\}, \{12, 17\}, \{12, 18\}, \{13, 19\}, \{14, 15\}, \{14, 17\}, \{15, 20\},$   
 $\{16, 21\}, \{17, 22\}, \{18, 19\}, \{18, 20\}, \{19, 22\}, \{20, 21\}, \{21, 22\}\}$  and  
 L20:  $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{3, 7\}, \{3, 8\}, \{4, 9\}, \{4, 10\},$   
 $\{5, 7\}, \{5, 9\}, \{6, 11\}, \{6, 12\}, \{7, 10\}, \{8, 13\}, \{8, 14\}, \{9, 15\}, \{10, 16\},$   
 $\{11, 13\}, \{11, 17\}, \{12, 16\}, \{12, 18\}, \{13, 19\}, \{14, 15\}, \{14, 18\}, \{15, 20\},$   
 $\{16, 21\}, \{17, 20\}, \{17, 22\}, \{18, 22\}, \{19, 21\}, \{19, 22\}, \{20, 21\}\}.$

The first two correspond to the Loupekine snarks #1 and #2 as supplied with the MATHEMATICA system, [24]. The remaining edge sets are precisely those of the first eighteen 22-vertex snarks in the Royale's list [26].

**Lemma 5.1** *There exist designs of order 22, 34, 55 and 88 for each of the twenty snarks on 22 vertices.*

**Proof.** Let the vertex set of  $K_{22}$  be  $Z_{21} \cup \{\infty\}$ . The decompositions consist of

$(\infty, 13, 0, 7, 3, 14, 18, 12, 4, 17, 5, 2, 11, 15, 6, 10, 8, 1, 9, 19, 16, 20)_{L1},$   
 $(\infty, 5, 4, 1, 13, 7, 18, 11, 17, 6, 0, 15, 3, 19, 16, 20, 12, 8, 2, 10, 14, 9)_{L2},$   
 $(\infty, 13, 3, 5, 17, 8, 2, 7, 19, 18, 11, 20, 6, 14, 1, 15, 16, 12, 0, 4, 10, 9)_{L3},$   
 $(\infty, 9, 10, 11, 4, 14, 13, 12, 18, 3, 15, 16, 8, 0, 6, 17, 20, 1, 19, 2, 7, 5)_{L4},$   
 $(\infty, 9, 5, 13, 19, 3, 18, 16, 14, 1, 8, 20, 12, 15, 11, 6, 0, 10, 7, 2, 17, 4)_{L5},$   
 $(\infty, 16, 18, 2, 9, 0, 12, 10, 14, 20, 4, 3, 17, 6, 19, 1, 15, 11, 5, 7, 8, 13)_{L6},$

$(\infty, 18, 10, 20, 16, 13, 7, 11, 2, 14, 9, 1, 19, 3, 5, 17, 8, 12, 6, 4, 15, 0)_{L7},$   
 $(\infty, 20, 19, 0, 2, 4, 12, 6, 14, 9, 15, 11, 3, 8, 17, 16, 18, 7, 13, 10, 1, 5)_{L8},$   
 $(\infty, 1, 14, 12, 15, 19, 16, 13, 4, 10, 7, 2, 17, 9, 20, 3, 11, 5, 0, 8, 6, 18)_{L9},$   
 $(\infty, 12, 2, 13, 14, 1, 11, 7, 16, 15, 6, 5, 20, 3, 9, 18, 19, 0, 8, 17, 10, 4)_{L10},$   
 $(\infty, 14, 16, 6, 15, 9, 0, 18, 8, 13, 5, 12, 19, 3, 4, 2, 17, 7, 20, 1, 10, 11)_{L11},$   
 $(\infty, 9, 20, 1, 19, 10, 11, 0, 3, 7, 17, 13, 14, 15, 12, 8, 4, 16, 2, 18, 6, 5)_{L12},$   
 $(\infty, 18, 17, 7, 8, 20, 11, 16, 13, 3, 15, 10, 19, 1, 2, 14, 5, 6, 12, 4, 0, 9)_{L13},$   
 $(\infty, 12, 10, 2, 4, 20, 0, 7, 16, 9, 5, 3, 8, 1, 6, 18, 14, 17, 19, 13, 11, 15)_{L14},$   
 $(\infty, 9, 10, 17, 4, 1, 3, 13, 16, 7, 20, 5, 19, 11, 18, 8, 2, 0, 6, 15, 12, 14)_{L15},$   
 $(\infty, 20, 9, 19, 6, 2, 7, 1, 8, 4, 10, 16, 15, 3, 0, 5, 14, 12, 13, 11, 18, 17)_{L16},$   
 $(\infty, 11, 19, 9, 4, 10, 13, 16, 17, 3, 12, 5, 20, 7, 8, 2, 6, 1, 18, 15, 0, 14)_{L17},$   
 $(\infty, 6, 13, 17, 4, 2, 20, 14, 7, 11, 5, 12, 1, 0, 16, 3, 18, 19, 9, 15, 8, 10)_{L18},$   
 $(\infty, 1, 3, 14, 15, 16, 7, 8, 12, 11, 19, 4, 0, 13, 18, 20, 2, 17, 10, 6, 5, 9)_{L19}$

and

$(\infty, 6, 7, 14, 20, 17, 19, 11, 12, 16, 8, 1, 5, 4, 10, 0, 3, 15, 9, 2, 13, 18)_{L20}$   
 under the action of the mapping  $\infty \mapsto \infty, x \mapsto x + 3 \pmod{21}$ .

Let the vertex set of  $K_{34}$  be  $Z_{34}$ . The decompositions consist of

$(7, 15, 32, 21, 17, 22, 10, 16, 3, 27, 1, 30, 0, 14, 19, 12, 9, 26, 20, 11, 18, 33)_{L1},$   
 $(14, 27, 31, 16, 28, 29, 11, 18, 10, 8, 20, 25, 22, 7, 26, 17, 5, 12, 33, 9, 23, 0)_{L2},$   
 $(17, 10, 11, 28, 20, 25, 23, 16, 26, 8, 22, 4, 3, 1, 6, 12, 7, 15, 31, 2, 33, 5)_{L3},$   
 $(5, 20, 21, 7, 30, 23, 32, 31, 1, 12, 27, 22, 14, 13, 0, 19, 26, 25, 6, 33, 8, 10)_{L4},$   
 $(15, 8, 2, 30, 1, 12, 25, 14, 4, 10, 21, 13, 5, 9, 0, 33, 23, 26, 28, 16, 31, 27)_{L5},$   
 $(16, 10, 25, 2, 32, 11, 6, 0, 19, 29, 15, 24, 31, 12, 5, 23, 9, 18, 1, 8, 3, 20)_{L6},$   
 $(20, 5, 19, 4, 23, 18, 27, 21, 0, 10, 11, 7, 16, 12, 24, 1, 25, 13, 14, 22, 3, 6)_{L7},$   
 $(20, 27, 24, 22, 19, 31, 25, 9, 1, 21, 7, 32, 12, 15, 6, 5, 26, 28, 17, 18, 10, 0)_{L8},$   
 $(1, 20, 2, 3, 12, 7, 21, 0, 8, 23, 10, 27, 24, 30, 31, 26, 18, 17, 19, 33, 9, 6)_{L9},$   
 $(20, 15, 31, 12, 9, 19, 23, 10, 30, 26, 7, 4, 6, 13, 25, 5, 29, 0, 22, 3, 32, 24)_{L10},$   
 $(9, 10, 24, 7, 31, 21, 2, 17, 27, 16, 20, 32, 12, 14, 26, 15, 19, 4, 5, 0, 3, 13)_{L11},$   
 $(6, 26, 29, 3, 28, 25, 23, 22, 11, 21, 15, 13, 31, 17, 12, 2, 8, 18, 32, 30, 20, 33)_{L12},$   
 $(15, 28, 14, 17, 12, 16, 23, 24, 27, 18, 22, 7, 4, 11, 21, 19, 0, 33, 5, 25, 26, 8)_{L13},$

$(26, 19, 15, 20, 7, 25, 6, 8, 4, 10, 21, 30, 16, 2, 32, 17, 11, 13, 33, 12, 31, 3)_{L14},$   
 $(29, 11, 20, 31, 26, 14, 16, 3, 4, 21, 25, 6, 8, 22, 0, 12, 5, 9, 13, 32, 19, 1)_{L15},$   
 $(22, 13, 26, 7, 17, 11, 3, 32, 0, 28, 16, 18, 24, 23, 4, 21, 6, 25, 8, 1, 9, 29)_{L16},$   
 $(3, 25, 8, 5, 24, 16, 12, 29, 13, 2, 23, 0, 19, 27, 17, 4, 6, 14, 20, 31, 11, 18)_{L17},$   
 $(4, 10, 18, 31, 23, 26, 25, 13, 3, 22, 0, 24, 21, 17, 32, 12, 7, 5, 20, 28, 11, 29)_{L18},$   
 $(17, 24, 23, 29, 10, 26, 14, 27, 32, 4, 15, 13, 8, 22, 16, 12, 5, 18, 19, 31, 11, 21)_{L19}$

and

$(4, 21, 25, 30, 10, 9, 23, 26, 28, 3, 18, 13, 12, 2, 29, 32, 14, 24, 7, 11, 1, 33)_{L20}$   
 under the action of the mapping  $x \mapsto x + 2 \pmod{34}$ .

Let the vertex set of  $K_{55}$  be  $Z_{54} \cup \{\infty\}$ . The decompositions consist of

$(\infty, 44, 47, 17, 13, 50, 10, 6, 23, 22, 1,$   
 $4, 3, 8, 31, 15, 53, 33, 45, 5, 24, 20)_{L1},$   
 $(36, 44, 29, 1, 7, 27, 43, 18, 6, 40, 23,$   
 $15, 31, 48, 30, 2, 37, 53, 0, 9, 19, 26)_{L1},$   
 $(49, 18, 23, 47, 35, 25, 42, 9, 39, 34, 16,$   
 $43, 20, 50, 24, 44, 26, 22, 53, 46, 36, 27)_{L1},$   
 $(11, 33, 24, 8, 41, 19, 14, 26, 6, 20, 15,$   
 $9, 44, 25, 0, 52, 27, 16, 45, 4, 29, 40)_{L1},$   
 $(2, 15, 46, 37, 34, 1, 4, 21, 42, 10, 29,$   
 $53, 35, 40, \infty, 16, 25, 7, 6, 38, 33, 0)_{L1},$   
 $(\infty, 22, 27, 6, 0, 23, 18, 11, 48, 49, 17,$   
 $2, 39, 44, 28, 25, 15, 26, 36, 50, 38, 24)_{L2},$   
 $(40, 52, 30, 5, 51, 6, 25, 37, 1, 34, 26,$   
 $10, 9, 48, 28, 3, 8, 14, 50, 49, 47, 23)_{L2},$   
 $(19, 21, 17, 43, 26, 46, 27, 14, 13, 7, 0,$   
 $36, 49, 24, 35, 33, 52, 3, 15, 9, 25, 28)_{L2},$   
 $(47, 20, 30, 42, 9, 23, 49, 29, 48, 53, 31,$   
 $16, 34, 35, 40, 28, 52, 15, 2, 0, 10, 51)_{L2},$   
 $(3, 2, 7, 11, 46, 31, 23, 16, 43, 9, 21,$   
 $50, 20, 24, 47, 18, 13, 44, \infty, 53, 41, 8)_{L2},$   
 $(\infty, 22, 35, 1, 39, 19, 49, 43, 16, 15, 14,$   
 $50, 45, 5, 29, 2, 18, 21, 10, 11, 42, 6)_{L3},$   
 $(44, 43, 12, 17, 28, 32, 29, 2, 10, 19, 26,$   
 $3, 4, 14, 47, 35, 53, 37, 31, 33, 1, 24)_{L3},$   
 $(29, 41, 42, 40, 33, 47, 45, 22, 31, 51, 20,$   
 $50, 52, 53, 39, 9, 30, 16, 4, 13, 18, 46)_{L3},$   
 $(21, 36, 0, 1, 30, 35, 11, 25, 26, 7, 43,$   
 $33, 9, 34, 42, 44, 12, 48, 51, 50, 24, 4)_{L3},$

$(19, 3, 42, 2, 29, 39, 24, \infty, 53, 33, 13,$   
 $50, 40, 46, 1, 10, 20, 23, 36, 52, 14, 38)_{L3},$   
 $(\infty, 0, 46, 14, 16, 53, 42, 47, 11, 18, 43,$   
 $44, 28, 24, 4, 2, 19, 21, 29, 32, 34, 22)_{L4},$   
 $(34, 46, 39, 7, 10, 13, 25, 19, 23, 36, 20,$   
 $38, 15, 0, 41, 47, 32, 3, 51, 30, 28, 45)_{L4},$   
 $(31, 39, 43, 21, 46, 6, 35, 14, 3, 47, 53,$   
 $37, 41, 24, 38, 26, 0, 8, 17, 12, 25, 32)_{L4},$   
 $(2, 46, 3, 52, 12, 49, 33, 19, 4, 31, 48,$   
 $37, 35, 11, 8, 28, 5, 9, 18, 27, 20, 23)_{L4},$   
 $(26, 31, 40, 32, 48, 24, 16, 9, 3, 50, 19,$   
 $36, 22, 45, 47, 27, 39, 11, 12, 7, \infty, 43)_{L4},$   
 $(\infty, 18, 43, 9, 29, 24, 44, 3, 48, 17, 10,$   
 $49, 20, 46, 41, 52, 15, 16, 27, 0, 21, 38)_{L5},$   
 $(53, 23, 24, 48, 37, 40, 1, 7, 41, 32, 25,$   
 $44, 9, 47, 0, 13, 43, 3, 27, 28, 21, 22)_{L5},$   
 $(8, 11, 38, 53, 2, 48, 19, 6, 4, 10, 51,$   
 $18, 30, 15, 21, 36, 32, 47, 39, 20, 45, 5)_{L5},$   
 $(32, 28, 26, 14, 45, 38, 41, 18, 0, 42, 22,$   
 $49, 31, 19, 12, 6, 37, 43, 34, 16, 50, 5)_{L5},$   
 $(36, 33, 32, 31, 2, 5, 53, \infty, 21, 16, 49,$   
 $17, 10, 23, 29, 52, 46, 25, 1, 50, 4, 51)_{L5},$   
 $(\infty, 25, 45, 14, 39, 48, 7, 53, 11, 32, 27,$   
 $22, 15, 2, 1, 36, 5, 49, 10, 41, 28, 51)_{L6},$   
 $(6, 19, 24, 15, 2, 48, 45, 44, 16, 14, 29,$   
 $47, 50, 23, 0, 10, 46, 26, 13, 3, 52, 38)_{L6},$   
 $(46, 7, 35, 27, 28, 3, 48, 17, 39, 19, 38,$   
 $16, 47, 0, 51, 14, 5, 31, 2, 49, 6, 30)_{L6},$   
 $(43, 29, 19, 50, 4, 23, 12, 32, 1, 36, 30,$   
 $44, 5, 53, 27, 0, 38, 31, 49, 21, 45, 37)_{L6},$   
 $(33, 27, 4, 30, 23, 10, 0, 26, \infty, 41, 22,$   
 $24, 49, 52, 40, 12, 50, 29, 21, 7, 16, 28)_{L6},$   
 $(\infty, 51, 48, 41, 42, 30, 33, 53, 12, 6, 49,$   
 $4, 19, 8, 37, 52, 31, 32, 46, 43, 16, 44)_{L7},$   
 $(8, 9, 4, 37, 48, 25, 14, 45, 0, 29, 41,$   
 $36, 26, 13, 23, 38, 17, 12, 39, 15, 52, 16)_{L7},$   
 $(53, 13, 2, 42, 8, 43, 35, 39, 22, 0, 25,$   
 $31, 47, 10, 40, 24, 9, 23, 28, 38, 20, 21)_{L7},$   
 $(17, 22, 10, 49, 35, 30, 42, 40, 26, 27, 20,$   
 $4, 23, 51, 47, 13, 6, 33, 39, 38, 32, 3)_{L7},$

$(51, 37, 43, 3, 47, 11, 49, 44, 29, 30, 12,$   
 $\infty, 14, 33, 4, 20, 45, 31, 34, 28, 5, 36)_{L7},$   
 $(\infty, 31, 12, 9, 30, 14, 52, 25, 15, 41, 3,$   
 $23, 33, 5, 35, 2, 21, 17, 19, 43, 28, 20)_{L8},$   
 $(36, 52, 8, 51, 5, 15, 42, 18, 44, 47, 50,$   
 $45, 6, 30, 49, 17, 40, 14, 27, 7, 12, 37)_{L8},$   
 $(47, 34, 8, 23, 24, 26, 28, 30, 1, 12, 13,$   
 $33, 49, 19, 45, 4, 42, 3, 15, 32, 35, 46)_{L8},$   
 $(31, 17, 42, 6, 48, 7, 50, 24, 47, 12, 28,$   
 $10, 33, 2, 4, 29, 52, 14, 43, 20, 25, 46)_{L8},$   
 $(24, 53, 5, 15, 34, 20, 32, 16, 29, 33, 3,$   
 $41, 26, \infty, 37, 4, 50, 22, 31, 51, 25, 28)_{L8},$   
 $(\infty, 12, 32, 10, 16, 5, 7, 13, 14, 33, 37,$   
 $19, 52, 31, 11, 42, 36, 40, 25, 6, 20, 3)_{L9},$   
 $(52, 41, 9, 44, 53, 39, 37, 5, 47, 45, 43,$   
 $50, 6, 16, 14, 12, 18, 11, 38, 48, 21, 3)_{L9},$   
 $(51, 0, 4, 8, 13, 30, 23, 48, 2, 50, 22,$   
 $19, 31, 32, 33, 46, 26, 52, 36, 11, 39, 47)_{L9},$   
 $(44, 53, 26, 15, 23, 46, 40, 52, 49, 51, 6,$   
 $13, 8, 3, 30, 38, 29, 7, 47, 41, 2, 22)_{L9},$   
 $(41, 26, 43, 27, 23, 46, 42, 13, 37, 25, 45,$   
 $16, \infty, 18, 22, 40, 53, 33, 12, 3, 24, 21)_{L9},$   
 $(\infty, 22, 38, 18, 50, 47, 51, 44, 30, 49, 29,$   
 $6, 5, 48, 21, 39, 20, 3, 4, 1, 16, 23)_{L10},$   
 $(21, 7, 6, 4, 44, 48, 15, 52, 41, 40, 11,$   
 $20, 39, 35, 37, 34, 0, 17, 25, 13, 26, 22)_{L10},$   
 $(33, 4, 21, 44, 36, 51, 47, 40, 34, 11, 24,$   
 $49, 41, 1, 52, 27, 29, 8, 43, 18, 20, 7)_{L10},$   
 $(10, 4, 24, 40, 11, 47, 18, 2, 19, 31, 42,$   
 $37, 20, 46, 22, 17, 13, 32, 44, 15, 21, 48)_{L10},$   
 $(50, 23, 27, 20, 42, 47, 37, 51, 12, 25, 48,$   
 $21, 32, 33, 49, 35, 18, 7, 15, 26, 34, \infty)_{L10},$   
 $(\infty, 18, 28, 31, 24, 29, 12, 2, 23, 34, 7,$   
 $4, 19, 14, 45, 47, 6, 50, 40, 8, 30, 44)_{L11},$   
 $(37, 44, 11, 15, 28, 43, 10, 25, 16, 18, 4,$   
 $31, 2, 47, 6, 22, 48, 20, 39, 45, 17, 21)_{L11},$   
 $(38, 41, 40, 17, 27, 21, 36, 30, 7, 32, 12,$   
 $52, 13, 29, 1, 31, 3, 10, 20, 25, 0, 51)_{L11},$   
 $(39, 25, 28, 43, 7, 18, 42, 31, 41, 14, 33,$   
 $20, 36, 0, 45, 35, 3, 1, 29, 10, 47, 27)_{L11},$

(25, 38, 52, 33, 48, 3, 22, 18, 41, 44, 53,  
 35, 5, 11, 40, 26,  $\infty$ , 20, 34, 15, 2, 27)<sub>L11</sub>,  
 ( $\infty$ , 6, 44, 19, 2, 13, 32, 31, 4, 16, 29,  
 3, 10, 45, 53, 20, 7, 36, 48, 34, 38, 35)<sub>L12</sub>,  
 (43, 1, 12, 23, 8, 37, 52, 15, 31, 38, 24,  
 7, 35, 44, 25, 17, 39, 46, 11, 16, 34, 4)<sub>L12</sub>,  
 (53, 24, 20, 52, 43, 5, 9, 40, 11, 30, 48,  
 50, 6, 8, 31, 41, 13, 26, 15, 21, 45, 51)<sub>L12</sub>,  
 (30, 25, 19, 45, 26, 40, 21, 13, 29, 53, 39,  
 49, 2, 28, 34, 52, 20, 10, 15, 18, 36, 6)<sub>L12</sub>,  
 (10, 3, 5, 41, 9, 18, 8, 45, 51, 39, 26,  
 12, 24,  $\infty$ , 7, 35, 50, 36, 53, 28, 47, 6)<sub>L12</sub>,  
 ( $\infty$ , 14, 31, 11, 47, 5, 53, 17, 30, 40, 29,  
 8, 52, 51, 18, 13, 9, 38, 6, 16, 39, 22)<sub>L13</sub>,  
 (45, 37, 32, 17, 41, 49, 5, 19, 30, 9, 29,  
 10, 8, 40, 27, 1, 4, 36, 33, 11, 39, 2)<sub>L13</sub>,  
 (28, 33, 23, 44, 47, 8, 3, 25, 48, 13, 30,  
 26, 27, 14, 50, 32, 15, 29, 49, 36, 24, 18)<sub>L13</sub>,  
 (37, 7, 4, 11, 53, 27, 46, 40, 50, 42, 32,  
 49, 13, 17, 34, 9, 20, 10, 1, 44, 36, 19)<sub>L13</sub>,  
 (17, 16, 34, 40, 39, 36, 9, 15, 0, 10, 6,  
 48, 8, 7, 31,  $\infty$ , 27, 20, 1, 18, 46, 12)<sub>L13</sub>,  
 ( $\infty$ , 4, 15, 47, 26, 44, 31, 40, 52, 21, 38,  
 42, 41, 51, 35, 9, 22, 34, 2, 27, 10, 48)<sub>L14</sub>,  
 (2, 31, 52, 21, 8, 28, 3, 42, 7, 53, 19,  
 9, 4, 41, 13, 16, 32, 6, 22, 49, 5, 24)<sub>L14</sub>,  
 (7, 52, 17, 15, 8, 41, 23, 38, 12, 18, 21,  
 32, 2, 34, 25, 3, 20, 19, 47, 30, 36, 6)<sub>L14</sub>,  
 (44, 12, 41, 45, 13, 18, 20, 3, 2, 30, 37,  
 38, 34, 40, 47, 14, 25, 1, 42, 49, 29, 51)<sub>L14</sub>,  
 (36, 29, 11, 27, 30, 52, 53, 35, 42, 51, 32,  
 28, 19,  $\infty$ , 45, 24, 41, 49, 7, 33, 13, 22)<sub>L14</sub>,  
 ( $\infty$ , 38, 36, 47, 0, 7, 29, 10, 14, 43, 20,  
 45, 39, 2, 22, 15, 25, 37, 26, 8, 51, 53)<sub>L15</sub>,  
 (50, 51, 9, 30, 23, 5, 29, 48, 0, 12, 52,  
 13, 36, 3, 10, 16, 47, 15, 26, 32, 35, 25)<sub>L15</sub>,  
 (23, 33, 47, 41, 34, 42, 32, 51, 27, 28, 37,  
 10, 40, 46, 7, 36, 15, 3, 17, 18, 19, 49)<sub>L15</sub>,  
 (1, 29, 44, 36, 19, 24, 28, 9, 14, 8, 20,  
 34, 15, 17, 18, 37, 4, 45, 6, 52, 31, 13)<sub>L15</sub>,



$(48, 50, 21, 35, 31, 49, \infty, 36, 13, 52, 42,$   
 $20, 17, 28, 47, 51, 41, 40, 16, 8, 44, 0)_{L15},$   
 $(\infty, 29, 39, 8, 3, 26, 53, 30, 35, 23, 18,$   
 $45, 43, 19, 16, 11, 34, 4, 47, 13, 14, 46)_{L16},$   
 $(20, 42, 28, 29, 23, 10, 19, 27, 21, 38, 48,$   
 $14, 2, 39, 1, 13, 8, 45, 9, 24, 31, 6)_{L16},$   
 $(35, 14, 41, 53, 9, 16, 2, 19, 43, 15, 5,$   
 $24, 18, 23, 22, 36, 28, 32, 52, 42, 11, 31)_{L16},$   
 $(11, 53, 12, 45, 36, 22, 27, 42, 33, 1, 16,$   
 $38, 29, 39, 2, 21, 10, 37, 0, 26, 25, 40)_{L16},$   
 $(47, 49, 52, 36, 46, 1, 10, 25, 9, \infty, 50,$   
 $30, 27, 40, 13, 0, 24, 19, 8, 51, 26, 20)_{L16},$   
 $(\infty, 41, 44, 52, 25, 0, 53, 13, 15, 29, 51,$   
 $40, 14, 39, 35, 37, 23, 22, 18, 26, 19, 48)_{L17},$   
 $(40, 38, 48, 16, 36, 41, 31, 11, 37, 18, 19,$   
 $51, 22, 32, 30, 4, 8, 45, 49, 25, 33, 29)_{L17},$   
 $(12, 17, 45, 43, 52, 13, 18, 53, 39, 32, 38,$   
 $6, 42, 46, 10, 0, 25, 3, 36, 27, 21, 35)_{L17},$   
 $(21, 23, 12, 18, 39, 13, 5, 41, 8, 38, 28,$   
 $22, 15, 44, 6, 32, 1, 7, 4, 14, 25, 11)_{L17},$   
 $(12, 35, 48, 34, 8, 47, 33, \infty, 20, 39, 15,$   
 $29, 31, 46, 44, 50, 38, 22, 19, 45, 10, 16)_{L17},$   
 $(\infty, 46, 45, 0, 38, 4, 11, 9, 31, 43, 41,$   
 $30, 48, 33, 6, 39, 26, 20, 1, 51, 53, 37)_{L18},$   
 $(6, 38, 36, 42, 5, 41, 20, 53, 40, 29, 15,$   
 $9, 7, 0, 45, 21, 22, 52, 37, 28, 1, 35)_{L18},$   
 $(0, 10, 4, 29, 43, 46, 38, 42, 47, 52, 2,$   
 $37, 1, 15, 36, 19, 27, 31, 5, 17, 21, 28)_{L18},$   
 $(21, 50, 46, 0, 32, 26, 6, 41, 51, 1, 39,$   
 $27, 36, 18, 28, 4, 38, 48, 3, 2, 8, 40)_{L18},$   
 $(21, 38, 49, 17, 7, 36, 19, 22, 20, 16, 30,$   
 $50, 37, 31, 8, 10, 41, 44, \infty, 23, 35, 47)_{L18},$   
 $(\infty, 11, 30, 8, 34, 43, 33, 50, 2, 7, 4,$   
 $1, 37, 42, 49, 5, 44, 26, 48, 14, 10, 13)_{L19},$   
 $(9, 13, 49, 18, 36, 43, 32, 5, 31, 2, 28,$   
 $17, 44, 3, 25, 42, 14, 6, 10, 24, 21, 27)_{L19},$   
 $(45, 33, 8, 25, 12, 47, 27, 41, 28, 5, 53,$   
 $10, 49, 48, 17, 9, 40, 29, 4, 32, 34, 52)_{L19},$   
 $(26, 51, 17, 44, 10, 20, 36, 35, 0, 34, 28,$   
 $29, 6, 14, 41, 39, 9, 3, 30, 11, 23, 24)_{L19},$

(50, 24, 10, 1, 36, 49, 45, 30, 41, 37, 51,  
22, 13, 47, 34, 27, 31, 15, 46, 8, 9,  $\infty$ )<sub>L19</sub>

and

( $\infty$ , 46, 0, 7, 5, 26, 32, 37, 45, 29, 51,  
28, 12, 34, 18, 14, 33, 22, 10, 40, 9, 1)<sub>L20</sub>,  
(33, 27, 21, 35, 14, 10, 49, 34, 18, 2, 46,  
31, 19, 17, 8, 32, 23, 25, 22, 51, 37, 11)<sub>L20</sub>,  
(45, 26, 43, 0, 9, 16, 34, 19, 5, 4, 11,  
27, 24, 39, 15, 20, 13, 30, 48, 25, 6, 50)<sub>L20</sub>,  
(32, 1, 26, 4, 11, 42, 8, 37, 30, 9, 14,  
49, 53, 6, 16, 13, 43, 29, 47, 51, 17, 0)<sub>L20</sub>,  
(49, 8, 48, 23, 50, 40, 27, 0, 4, 24, 47,  
44, 15, 21, 29, 6, 26,  $\infty$ , 53, 18, 52, 17)<sub>L20</sub>

under the action of the mapping  $\infty \mapsto \infty$ ,  $x \mapsto x + 6 \pmod{54}$ .

Let the vertex set of  $K_{67}$  be  $Z_{67}$ . The decompositions consist of

(40, 6, 35, 50, 31, 10, 39, 16, 8, 2, 49,  
9, 63, 55, 44, 0, 19, 7, 24, 64, 22, 29)<sub>L1</sub>,

(26, 27, 60, 55, 40, 38, 28, 31, 17, 47, 32,  
48, 23, 29, 43, 3, 7, 30, 13, 0, 24, 5)<sub>L2</sub>,

(21, 61, 36, 9, 13, 31, 12, 22, 44, 25, 54,  
26, 52, 32, 59, 3, 18, 11, 41, 20, 0, 28)<sub>L3</sub>,

(25, 53, 45, 54, 41, 55, 60, 58, 27, 37, 63,  
23, 16, 59, 64, 2, 0, 18, 7, 6, 36, 43)<sub>L4</sub>,

(56, 58, 43, 0, 40, 63, 44, 33, 25, 24, 47,  
5, 36, 19, 15, 12, 3, 27, 61, 1, 32, 39)<sub>L5</sub>,

(7, 22, 25, 21, 15, 30, 56, 27, 42, 8, 2,  
45, 43, 51, 19, 23, 31, 33, 32, 28, 66, 57)<sub>L6</sub>,

(54, 43, 35, 10, 27, 47, 21, 2, 30, 41, 26,  
9, 15, 48, 14, 0, 16, 13, 55, 56, 46, 38)<sub>L7</sub>,

(64, 28, 39, 15, 37, 2, 60, 1, 27, 9, 47,  
33, 16, 8, 41, 0, 21, 4, 58, 43, 54, 7)<sub>L8</sub>,

(59, 22, 20, 61, 11, 7, 12, 16, 37, 54, 39,  
57, 32, 47, 33, 6, 51, 46, 9, 52, 64, 29)<sub>L9</sub>,

(43, 52, 64, 50, 34, 57, 20, 45, 35, 24, 23,  
49, 51, 5, 59, 6, 4, 0, 1, 13, 2, 28)<sub>L10</sub>,

(14, 7, 59, 30, 34, 57, 3, 2, 55, 62, 60,  
53, 49, 48, 31, 1, 19, 5, 50, 44, 45, 17)<sub>L11</sub>,

(30, 7, 24, 17, 59, 45, 40, 3, 35, 47, 56,  
20, 61, 50, 11, 10, 21, 66, 46, 23, 43, 38)<sub>L12</sub>,

(32, 28, 18, 61, 52, 2, 45, 31, 27, 44, 49,  
24, 23, 59, 47, 3, 8, 62, 57, 11, 51, 20)<sub>L13</sub>,

(46, 55, 61, 13, 41, 60, 48, 30, 2, 3, 59,  
28, 22, 43, 45, 1, 11, 40, 20, 21, 57, 17)<sub>L14</sub>,

(11, 61, 47, 0, 40, 20, 59, 49, 37, 52, 55,  
10, 62, 3, 53, 2, 1, 24, 42, 26, 64, 44)<sub>L15</sub>,

(0, 28, 63, 57, 2, 35, 50, 48, 33, 25, 65,  
55, 45, 31, 37, 3, 21, 22, 10, 38, 30, 32)<sub>L16</sub>,

(27, 56, 35, 0, 60, 21, 63, 20, 53, 18, 11,  
15, 40, 2, 51, 10, 34, 23, 3, 57, 48, 5)<sub>L17</sub>,

(10, 3, 48, 20, 50, 40, 17, 23, 59, 19, 44,  
55, 39, 31, 45, 0, 34, 1, 13, 28, 24, 2)<sub>L18</sub>,

(45, 28, 35, 41, 52, 64, 20, 1, 57, 60, 50,  
51, 24, 23, 48, 4, 21, 54, 53, 42, 22, 14)<sub>L19</sub>

and

(54, 44, 27, 10, 41, 9, 52, 7, 36, 48, 17,  
42, 38, 23, 45, 5, 11, 40, 26, 30, 12, 39)<sub>L20</sub>

under the action of the mapping  $x \mapsto x + 1 \pmod{67}$ .

Let the vertex set of  $K_{88}$  be  $Z_{87} \cup \{\infty\}$ . The decompositions consist of

(9, 84, 83, 29, 63, 44, 22, 46, 14, 42, 86,  
52, 1, 72, 64, 74, 68, 45, 30, 55, 75, 54)<sub>L1</sub>,

(27, 74, 62, 24, 79, 40, 22, 50, 33, 7, 17,  
11, 65, 73, 9, 86, 55, 20, 15, 66, 49, 43)<sub>L1</sub>,

(58, 59, 55, 30, 40, 31, 1, 71, 32, 46, 47,  
57, 82, 0, 29, 39, 81, 20, 56, 78, 86, 43)<sub>L1</sub>,

(73, 2, 24, 6, 12, 52, 10, 4, 13, 83, 38,  
 $\infty$ , 78, 27, 69, 17, 56, 42, 49, 18, 22, 20)<sub>L1</sub>,

(20, 47, 84, 50, 57, 16, 22, 26, 37, 9, 46,  
24, 76, 74, 80, 7, 30, 21, 56, 54, 8, 49)<sub>L2</sub>,

(86, 9, 17, 52, 40, 73, 21, 62, 84, 24, 35,  
47, 71, 82, 1, 30, 57, 27, 59, 19, 2, 55)<sub>L2</sub>,

(33, 53, 71, 47, 62, 18, 0, 21, 82, 45, 55,  
75, 57, 42, 85, 13, 17, 16, 26, 83, 19, 9)<sub>L2</sub>,

(41, 15, 1, 83, 19, 72, 10, 4, 0, 42, 27,  
6, 8, 50, 85, 13, 7, 64, 32, 65,  $\infty$ , 70)<sub>L2</sub>,

$(\infty, 34, 53, 39, 7, 75, 40, 0, 14, 24, 42,$   
 $5, 82, 74, 59, 36, 76, 63, 23, 21, 84, 20)_{L3},$   
 $(10, 45, 61, 33, 18, 7, 31, 79, 15, 41, 39,$   
 $57, 86, 19, 80, 1, 63, 47, 17, 4, 48, 64)_{L3},$   
 $(0, 45, 57, 34, 78, 41, 67, 71, 26, 80, 35,$   
 $50, 25, 54, 79, 85, 13, 56, 40, 75, 82, 11)_{L3},$   
 $(34, 18, 71, 83, 13, 55, 4, 76, 77, 79, 30,$   
 $3, 23, 21, 36, 22, 14, 86, 8, 29, 39, 26)_{L3},$   
 $(\infty, 86, 55, 54, 25, 0, 37, 39, 14, 10, 31,$   
 $11, 81, 32, 78, 36, 29, 59, 51, 49, 64, 53)_{L4},$   
 $(25, 62, 39, 46, 44, 32, 15, 19, 80, 5, 0,$   
 $16, 52, 42, 38, 48, 75, 8, 40, 73, 54, 70)_{L4},$   
 $(60, 65, 9, 56, 81, 17, 78, 10, 2, 85, 39,$   
 $64, 70, 8, 6, 15, 46, 52, 72, 29, 61, 7)_{L4},$   
 $(63, 54, 28, 44, 37, 64, 68, 3, 0, 2, 60,$   
 $15, 26, 73, 7, 8, 22, 84, 80, 86, 11, 71)_{L4},$   
 $(\infty, 64, 84, 17, 5, 6, 14, 38, 39, 69, 27,$   
 $49, 79, 33, 32, 53, 37, 80, 47, 21, 60, 66)_{L5},$   
 $(4, 62, 37, 17, 29, 24, 53, 64, 56, 33, 49,$   
 $84, 9, 7, 47, 80, 72, 0, 61, 26, 73, 79)_{L5},$   
 $(79, 56, 4, 64, 50, 68, 60, 39, 47, 13, 34,$   
 $63, 41, 11, 73, 10, 42, 16, 30, 8, 75, 53)_{L5},$   
 $(13, 10, 19, 72, 84, 35, 15, 80, 12, 46, 55,$   
 $77, 25, 42, 69, 27, 44, 61, 26, 31, 36, 37)_{L5},$   
 $(\infty, 14, 43, 66, 73, 3, 59, 61, 33, 11, 30,$   
 $39, 48, 12, 34, 38, 37, 60, 31, 62, 8, 68)_{L6},$   
 $(8, 61, 49, 39, 64, 12, 79, 13, 21, 78, 84,$   
 $71, 27, 18, 59, 45, 73, 43, 5, 82, 7, 68)_{L6},$   
 $(85, 40, 7, 75, 71, 62, 66, 20, 35, 82, 2,$   
 $3, 30, 39, 18, 70, 13, 68, 37, 80, 63, 15)_{L6},$   
 $(38, 44, 43, 37, 59, 5, 16, 26, 70, 6, 63,$   
 $71, 49, 23, 27, 36, 75, 11, 13, 20, 22, 80)_{L6},$   
 $(\infty, 85, 11, 75, 10, 47, 67, 13, 33, 22, 15,$   
 $62, 72, 58, 0, 80, 66, 6, 38, 86, 57, 64)_{L7},$   
 $(60, 64, 49, 42, 65, 32, 26, 5, 56, 4, 11,$   
 $29, 6, 70, 3, 9, 33, 57, 30, 52, 16, 14)_{L7},$   
 $(54, 8, 69, 2, 78, 19, 34, 79, 66, 65, 28,$   
 $36, 64, 82, 23, 4, 67, 25, 5, 51, 59, 45)_{L7},$   
 $(69, 30, 20, 59, 41, 67, 1, 50, 37, 65, 32,$   
 $7, 35, 79, 53, 61, 39, 66, 5, 58, 72, 42)_{L7},$

$(\infty, 80, 34, 66, 26, 54, 4, 85, 6, 37, 79,$   
 $45, 78, 75, 84, 11, 5, 30, 35, 51, 31, 86)_{L8},$   
 $(73, 84, 44, 86, 48, 82, 34, 66, 22, 40, 67,$   
 $30, 3, 9, 29, 7, 25, 32, 64, 11, 70, 85)_{L8},$   
 $(79, 76, 65, 15, 82, 80, 63, 71, 11, 67, 35,$   
 $62, 39, 74, 55, 28, 0, 77, 17, 38, 53, 33)_{L8},$   
 $(22, 81, 39, 14, 74, 35, 24, 30, 27, 42, 85,$   
 $53, 38, 57, 54, 36, 37, 13, 55, 44, 4, 11)_{L8},$   
 $(\infty, 35, 1, 27, 22, 82, 3, 18, 76, 51, 54,$   
 $64, 32, 79, 46, 26, 61, 10, 4, 20, 81, 31)_{L9},$   
 $(43, 5, 67, 18, 19, 6, 23, 21, 74, 38, 75,$   
 $78, 25, 20, 64, 62, 7, 41, 14, 84, 13, 80)_{L9},$   
 $(69, 17, 27, 39, 21, 16, 60, 15, 28, 14, 35,$   
 $65, 46, 62, 72, 45, 71, 54, 47, 20, 80, 1)_{L9},$   
 $(82, 43, 24, 37, 57, 35, 79, 33, 68, 14, 47,$   
 $64, 39, 65, 49, 58, 77, 2, 84, 48, 12, 18)_{L9},$   
 $(\infty, 37, 21, 44, 47, 23, 46, 52, 33, 76, 1,$   
 $58, 16, 14, 84, 28, 20, 60, 18, 77, 48, 73)_{L10},$   
 $(12, 52, 85, 60, 72, 2, 29, 74, 54, 55, 19,$   
 $21, 18, 44, 9, 73, 51, 25, 81, 38, 79, 36)_{L10},$   
 $(75, 8, 48, 53, 28, 60, 22, 47, 6, 41, 83,$   
 $7, 65, 27, 76, 45, 17, 20, 51, 11, 55, 77)_{L10},$   
 $(60, 72, 61, 69, 0, 1, 64, 3, 62, 23, 86,$   
 $83, 6, 65, 70, 58, 11, 25, 44, 73, 47, 49)_{L10},$   
 $(\infty, 24, 65, 76, 47, 8, 53, 56, 12, 0, 40,$   
 $29, 41, 72, 16, 32, 74, 18, 73, 44, 61, 11)_{L11},$   
 $(1, 15, 10, 9, 2, 47, 71, 29, 40, 66, 37,$   
 $72, 80, 13, 49, 83, 7, 42, 31, 54, 81, 48)_{L11},$   
 $(69, 6, 2, 19, 22, 58, 42, 48, 25, 14, 73,$   
 $7, 67, 45, 50, 41, 47, 63, 43, 72, 51, 16)_{L11},$   
 $(64, 63, 60, 53, 71, 31, 34, 41, 23, 36, 85,$   
 $32, 83, 81, 37, 18, 79, 80, 15, 30, 69, 58)_{L11},$   
 $(\infty, 57, 86, 85, 34, 23, 7, 54, 47, 4, 8,$   
 $40, 3, 43, 30, 62, 19, 51, 73, 14, 65, 31)_{L12},$   
 $(14, 78, 67, 41, 60, 19, 74, 49, 80, 24, 18,$   
 $9, 5, 17, 61, 82, 6, 28, 42, 21, 47, 33)_{L12},$   
 $(64, 72, 81, 15, 23, 42, 80, 27, 17, 19, 40,$   
 $48, 8, 35, 20, 59, 73, 53, 31, 10, 41, 66)_{L12},$   
 $(25, 75, 76, 78, 35, 30, 67, 0, 66, 68, 59,$   
 $85, 71, 60, 83, 24, 13, 73, 82, 84, 43, 32)_{L12},$

$(\infty, 12, 50, 67, 6, 41, 4, 32, 82, 70, 86,$   
 $85, 15, 20, 69, 39, 25, 51, 33, 34, 24, 30)_{L13},$   
 $(23, 62, 50, 70, 85, 77, 86, 66, 51, 56, 9,$   
 $15, 79, 27, 65, 41, 47, 55, 7, 73, 30, 16)_{L13},$   
 $(33, 80, 66, 29, 5, 78, 7, 32, 72, 49, 39,$   
 $34, 58, 17, 82, 40, 3, 51, 76, 86, 2, 45)_{L13},$   
 $(39, 62, 27, 31, 6, 25, 67, 17, 14, 9, 20,$   
 $37, 52, 15, 75, 64, 74, 71, 29, 47, 13, 40)_{L13},$   
 $(\infty, 7, 56, 60, 20, 0, 29, 19, 14, 45, 55,$   
 $62, 25, 28, 81, 80, 30, 35, 17, 31, 38, 76)_{L14},$   
 $(63, 40, 60, 80, 27, 29, 53, 81, 37, 84, 64,$   
 $32, 69, 46, 31, 61, 17, 47, 2, 45, 3, 70)_{L14},$   
 $(37, 59, 20, 56, 55, 78, 57, 60, 8, 44, 13,$   
 $9, 18, 38, 42, 4, 66, 70, 72, 16, 77, 27)_{L14},$   
 $(27, 80, 50, 36, 19, 13, 14, 5, 40, 16, 61,$   
 $85, 38, 69, 79, 54, 52, 9, 48, 30, 49, 68)_{L14},$   
 $(\infty, 13, 66, 80, 64, 48, 60, 14, 55, 69, 62,$   
 $76, 51, 50, 79, 58, 84, 39, 12, 41, 17, 72)_{L15},$   
 $(24, 50, 3, 55, 63, 72, 40, 21, 38, 68, 1,$   
 $2, 81, 70, 36, 43, 26, 49, 7, 6, 76, 9)_{L15},$   
 $(79, 33, 32, 52, 26, 17, 53, 77, 37, 30, 68,$   
 $74, 23, 31, 73, 16, 13, 20, 18, 25, 42, 4)_{L15},$   
 $(21, 17, 29, 70, 23, 57, 41, 44, 13, 51, 42,$   
 $11, 1, 86, 76, 3, 40, 20, 27, 78, 71, 73)_{L15},$   
 $(\infty, 81, 23, 67, 8, 52, 12, 13, 57, 6, 16,$   
 $19, 44, 70, 43, 41, 18, 3, 36, 35, 47, 63)_{L16},$   
 $(4, 13, 73, 76, 68, 69, 51, 71, 60, 55, 37,$   
 $30, 49, 86, 27, 67, 15, 2, 82, 29, 38, 40)_{L16},$   
 $(74, 6, 11, 62, 55, 83, 20, 68, 49, 17, 33,$   
 $30, 52, 8, 37, 78, 35, 80, 75, 85, 16, 21)_{L16},$   
 $(10, 85, 57, 11, 15, 6, 78, 16, 4, 42, 86,$   
 $49, 2, 18, 54, 47, 51, 35, 70, 53, 60, 32)_{L16},$   
 $(\infty, 30, 77, 40, 70, 64, 37, 31, 2, 66, 73,$   
 $29, 74, 24, 23, 14, 9, 41, 19, 3, 54, 16)_{L17},$   
 $(46, 56, 57, 50, 54, 61, 67, 24, 66, 60, 47,$   
 $83, 75, 20, 86, 35, 30, 3, 13, 73, 27, 28)_{L17},$   
 $(20, 47, 28, 60, 70, 18, 48, 79, 58, 25, 55,$   
 $68, 17, 71, 83, 16, 37, 6, 56, 26, 1, 84)_{L17},$   
 $(77, 69, 11, 61, 37, 68, 74, 30, 67, 78, 12,$   
 $54, 76, 39, 29, 47, 51, 3, 55, 56, 81, 73)_{L17},$

$(\infty, 44, 57, 31, 14, 13, 5, 33, 74, 60, 51,$   
 $47, 48, 6, 16, 56, 81, 45, 40, 65, 49, 19)_{L18},$   
 $(61, 62, 79, 63, 85, 33, 4, 29, 57, 27, 76,$   
 $71, 30, 25, 80, 36, 6, 12, 22, 59, 43, 83)_{L18},$   
 $(76, 4, 62, 8, 60, 24, 64, 17, 43, 27, 16,$   
 $45, 5, 3, 67, 11, 20, 53, 59, 66, 37, 79)_{L18},$   
 $(42, 24, 53, 12, 57, 76, 9, 35, 73, 85, 5,$   
 $68, 2, 81, 34, 25, 20, 32, 39, 17, 61, 7)_{L18},$   
 $(\infty, 77, 24, 61, 34, 16, 52, 20, 69, 58, 83,$   
 $86, 81, 29, 62, 71, 54, 57, 82, 35, 84, 7)_{L19},$   
 $(75, 21, 43, 80, 51, 70, 65, 28, 59, 46, 72,$   
 $74, 52, 19, 3, 85, 15, 67, 45, 48, 54, 27)_{L19},$   
 $(24, 60, 72, 65, 63, 81, 46, 83, 23, 10, 52,$   
 $38, 53, 48, 34, 47, 85, 18, 68, 86, 76, 43)_{L19},$   
 $(73, 52, 84, 17, 80, 79, 74, 6, 31, 56, 8,$   
 $38, 78, 59, 61, 72, 62, 39, 34, 15, 49, 26)_{L19}$

and

$(\infty, 62, 76, 45, 15, 13, 72, 25, 60, 66, 23,$   
 $28, 37, 29, 17, 75, 7, 83, 42, 64, 78, 31)_{L20},$   
 $(46, 45, 20, 59, 83, 12, 81, 0, 11, 13, 71,$   
 $82, 48, 55, 38, 57, 2, 53, 29, 75, 49, 60)_{L20},$   
 $(27, 14, 17, 5, 80, 76, 74, 70, 61, 29, 18,$   
 $28, 52, 3, 16, 30, 59, 21, 53, 22, 57, 31)_{L20},$   
 $(28, 5, 61, 35, 30, 16, 58, 12, 38, 79, 66,$   
 $8, 83, 65, 29, 63, 25, 60, 72, 34, 7, 48)_{L20}$

under the action of the mapping  $\infty \mapsto \infty, x \mapsto x + 3 \pmod{87}$ .  $\square$

**Lemma 5.2** *There exist decompositions of  $K_{33,33,33}$ ,  $K_{11,11,11,11}$ ,  $K_{22,22,22,55}$  and  $K_{22,22,22,22,22,55}$  into each of the twenty snarks on 22 vertices.*

**Proof.** Let the vertex set of  $K_{33,33,33}$  be  $Z_{99}$  partitioned according to residue classes modulo 3. The decompositions consist of

$(10, 29, 62, 52, 93, 55, 96, 73, 0, 35, 85,$   
 $77, 24, 19, 59, 3, 57, 88, 98, 31, 61, 18)_{L1},$   
 $(91, 6, 15, 46, 26, 16, 41, 63, 38, 84, 58,$   
 $54, 13, 56, 82, 11, 17, 48, 40, 78, 18, 43)_{L2},$   
 $(44, 12, 66, 16, 86, 88, 70, 26, 33, 96, 50,$   
 $31, 20, 94, 1, 0, 72, 36, 65, 71, 10, 79)_{L3},$   
 $(8, 24, 21, 75, 31, 83, 68, 97, 80, 37, 81,$   
 $2, 55, 71, 4, 3, 11, 63, 32, 84, 20, 52)_{L4},$

$(12, 40, 80, 64, 96, 57, 46, 25, 38, 68, 97,$   
 $69, 11, 16, 77, 5, 13, 8, 3, 70, 51, 91)_{L5},$   
 $(61, 56, 11, 50, 12, 63, 10, 88, 91, 79, 23,$   
 $62, 84, 78, 9, 25, 8, 13, 97, 5, 40, 59)_{L6},$   
 $(12, 19, 23, 50, 60, 5, 55, 4, 85, 90, 88,$   
 $32, 38, 41, 28, 6, 10, 18, 87, 70, 20, 74)_{L7},$   
 $(94, 6, 74, 89, 76, 20, 15, 22, 57, 55, 78,$   
 $77, 95, 85, 32, 10, 4, 72, 39, 28, 53, 56)_{L8},$   
 $(47, 12, 25, 88, 56, 2, 42, 90, 63, 50, 96,$   
 $10, 92, 20, 81, 1, 39, 64, 40, 93, 31, 17)_{L9},$   
 $(84, 71, 17, 2, 16, 61, 90, 9, 39, 1, 63,$   
 $47, 28, 53, 41, 13, 56, 33, 60, 91, 21, 44)_{L10},$   
 $(54, 11, 7, 91, 18, 97, 92, 45, 62, 81, 47,$   
 $74, 61, 46, 13, 6, 15, 29, 32, 33, 73, 37)_{L11},$   
 $(78, 70, 65, 94, 89, 66, 40, 24, 84, 74, 23,$   
 $55, 61, 60, 76, 0, 57, 29, 38, 88, 59, 64)_{L12},$   
 $(65, 84, 45, 22, 74, 70, 67, 73, 87, 96, 75,$   
 $69, 18, 64, 29, 7, 68, 71, 38, 13, 3, 21)_{L13},$   
 $(16, 42, 62, 59, 86, 58, 52, 12, 45, 27, 80,$   
 $56, 50, 85, 19, 4, 55, 57, 21, 0, 32, 20)_{L14},$   
 $(55, 83, 26, 87, 90, 52, 88, 39, 11, 44, 74,$   
 $18, 79, 93, 70, 16, 28, 29, 40, 59, 24, 54)_{L15},$   
 $(19, 89, 84, 24, 63, 36, 25, 7, 79, 35, 86,$   
 $44, 50, 54, 3, 0, 45, 78, 68, 95, 97, 31)_{L16},$   
 $(66, 52, 40, 82, 23, 48, 18, 65, 81, 35, 67,$   
 $94, 96, 32, 3, 47, 2, 58, 68, 91, 69, 34)_{L17},$   
 $(89, 1, 3, 4, 0, 27, 83, 73, 32, 63, 58,$   
 $37, 50, 48, 28, 65, 95, 98, 43, 81, 9, 46)_{L18},$   
 $(87, 44, 38, 89, 6, 36, 10, 25, 52, 75, 77,$   
 $65, 72, 92, 69, 55, 81, 34, 8, 59, 0, 1)_{L19}$

and

$(75, 20, 13, 97, 15, 96, 47, 9, 44, 54, 70,$   
 $32, 83, 56, 45, 4, 53, 42, 81, 61, 23, 22)_{L20}$

under the action of the mapping  $x \mapsto x + 1 \pmod{99}$ .

Let the vertex set of  $K_{11,11,11,11}$  be  $Z_{44}$  partitioned according to residue classes modulo 4. The decompositions consist of



(24, 35, 14, 13, 11, 41, 23, 4, 18, 5, 26,  
 17, 19, 0, 33, 1, 2, 7, 8, 27, 29, 6)<sub>L1</sub>,  
 (0, 1, 2, 4, 7, 13, 31, 6, 11, 22, 36,  
 25, 12, 35, 20, 34, 3, 30, 37, 16, 28, 42)<sub>L1</sub>  
 (26, 35, 23, 32, 39, 6, 1, 2, 24, 25, 42,  
 19, 28, 15, 33, 0, 5, 3, 21, 10, 31, 4)<sub>L2</sub>,  
 (0, 2, 5, 3, 1, 8, 24, 7, 4, 34, 43,  
 30, 13, 40, 33, 28, 17, 42, 27, 6, 22, 29)<sub>L2</sub>  
 (18, 28, 27, 41, 19, 35, 33, 8, 6, 14, 23,  
 9, 12, 25, 22, 0, 1, 3, 11, 7, 10, 16)<sub>L3</sub>,  
 (0, 2, 3, 5, 1, 4, 24, 22, 19, 30, 38,  
 29, 14, 8, 25, 12, 9, 21, 27, 35, 18, 28)<sub>L3</sub>,  
 (16, 31, 11, 38, 34, 13, 5, 22, 39, 8, 36,  
 9, 14, 6, 35, 0, 4, 12, 1, 21, 2, 23)<sub>L4</sub>,  
 (0, 3, 5, 6, 1, 9, 2, 35, 32, 23, 42,  
 4, 39, 25, 7, 31, 10, 36, 38, 28, 21, 17)<sub>L4</sub>,  
 (20, 22, 15, 11, 27, 12, 0, 25, 33, 26, 37,  
 5, 18, 31, 8, 2, 1, 6, 7, 39, 28, 21)<sub>L5</sub>,  
 (0, 1, 3, 6, 2, 4, 24, 30, 12, 43, 25,  
 22, 16, 5, 39, 17, 35, 15, 38, 13, 26, 8)<sub>L5</sub>,  
 (22, 21, 17, 11, 32, 7, 10, 36, 30, 29, 0,  
 19, 35, 33, 3, 1, 4, 6, 14, 27, 5, 24)<sub>L6</sub>,  
 (0, 2, 5, 6, 3, 7, 12, 14, 33, 20, 38,  
 31, 11, 39, 9, 36, 28, 32, 22, 29, 42, 1)<sub>L6</sub>,  
 (16, 42, 7, 14, 23, 13, 21, 32, 25, 33, 30,  
 15, 10, 3, 12, 0, 1, 4, 2, 6, 11, 9)<sub>L7</sub>,  
 (0, 1, 7, 10, 4, 6, 17, 37, 27, 43, 12,  
 32, 20, 18, 25, 2, 3, 38, 41, 31, 36, 23)<sub>L7</sub>,  
 (27, 14, 13, 22, 31, 28, 40, 26, 12, 4, 23,  
 17, 24, 35, 30, 1, 2, 5, 9, 7, 8, 0)<sub>L8</sub>,  
 (0, 3, 6, 7, 2, 5, 1, 15, 13, 33, 16,  
 30, 18, 28, 42, 43, 37, 9, 27, 35, 10, 12)<sub>L8</sub>,  
 (12, 11, 35, 5, 42, 14, 13, 33, 0, 39, 37,  
 2, 34, 9, 24, 6, 4, 7, 15, 20, 1, 32)<sub>L9</sub>,  
 (0, 1, 2, 3, 6, 15, 7, 8, 9, 29, 26,  
 12, 42, 24, 39, 4, 23, 19, 37, 34, 41, 10)<sub>L9</sub>,  
 (38, 28, 12, 11, 15, 18, 10, 17, 40, 4, 25,  
 21, 8, 37, 34, 0, 1, 2, 3, 23, 32, 29)<sub>L10</sub>,

$(0, 1, 3, 6, 7, 12, 8, 34, 9, 36, 30,$   
 $19, 21, 35, 15, 5, 11, 26, 2, 41, 18, 27)_{L10},$   
 $(39, 25, 4, 37, 35, 34, 26, 38, 22, 19, 23,$   
 $36, 28, 1, 8, 3, 2, 0, 5, 7, 13, 27)_{L11},$   
 $(0, 2, 3, 5, 1, 7, 6, 28, 24, 20, 34,$   
 $12, 17, 43, 9, 25, 10, 13, 22, 30, 31, 40)_{L11},$   
 $(5, 24, 42, 35, 27, 19, 37, 13, 16, 34, 22,$   
 $30, 2, 21, 32, 3, 4, 1, 12, 31, 9, 18)_{L12},$   
 $(0, 1, 2, 5, 3, 4, 21, 7, 26, 19, 33,$   
 $37, 28, 10, 40, 43, 20, 27, 36, 35, 6, 14)_{L12},$   
 $(7, 2, 9, 6, 8, 29, 19, 4, 21, 32, 18,$   
 $36, 13, 20, 38, 11, 14, 3, 1, 12, 22, 27)_{L13},$   
 $(0, 3, 7, 10, 1, 4, 12, 13, 24, 41, 27,$   
 $31, 43, 21, 22, 9, 6, 26, 14, 17, 36, 35)_{L13},$   
 $(11, 34, 20, 37, 39, 24, 41, 7, 36, 3, 42,$   
 $29, 25, 31, 13, 0, 2, 6, 14, 15, 16, 1)_{L14},$   
 $(0, 2, 6, 7, 3, 5, 28, 13, 10, 22, 40,$   
 $26, 24, 23, 9, 41, 36, 37, 35, 16, 18, 15)_{L14},$   
 $(36, 9, 23, 25, 31, 10, 29, 0, 12, 20, 41,$   
 $13, 14, 27, 42, 2, 3, 4, 1, 16, 8, 7)_{L15},$   
 $(0, 1, 2, 6, 7, 11, 9, 3, 33, 23, 22,$   
 $14, 26, 19, 40, 32, 28, 5, 10, 21, 39, 30)_{L15},$   
 $(18, 31, 33, 15, 13, 17, 12, 36, 14, 30, 42,$   
 $23, 5, 16, 20, 1, 0, 3, 2, 10, 38, 27)_{L16},$   
 $(0, 5, 7, 9, 10, 12, 21, 2, 31, 11, 37,$   
 $39, 33, 16, 1, 14, 8, 4, 27, 18, 3, 36)_{L16},$   
 $(1, 16, 18, 4, 17, 14, 12, 41, 22, 23, 39,$   
 $29, 11, 35, 30, 0, 3, 2, 5, 32, 8, 7)_{L17},$   
 $(0, 2, 9, 10, 5, 7, 16, 40, 28, 41, 18,$   
 $17, 31, 15, 6, 3, 11, 1, 14, 33, 38, 24)_{L17},$   
 $(17, 15, 31, 39, 8, 29, 2, 28, 42, 33, 22,$   
 $14, 3, 23, 24, 0, 5, 4, 6, 1, 9, 26)_{L18},$   
 $(0, 1, 5, 6, 2, 3, 32, 12, 20, 17, 14,$   
 $21, 23, 11, 33, 27, 40, 15, 28, 38, 26, 31)_{L18},$   
 $(24, 35, 27, 5, 6, 2, 0, 36, 7, 18, 21,$   
 $23, 14, 38, 8, 4, 9, 1, 11, 31, 13, 16)_{L19},$   
 $(0, 1, 5, 7, 2, 6, 11, 16, 25, 13, 43,$   
 $8, 26, 37, 34, 28, 27, 42, 9, 3, 22, 40)_{L19}$

and

(27, 37, 2, 16, 31, 40, 24, 15, 14, 6, 41,  
35, 12, 10, 25, 0, 3, 5, 1, 4, 18, 26)<sub>L20</sub>,  
(0, 2, 5, 9, 1, 3, 14, 12, 28, 7, 37,  
29, 39, 35, 38, 22, 19, 16, 24, 41, 11, 10)<sub>L20</sub>

under the action of the mapping  $x \mapsto x + 4 \pmod{44}$ .

Let the vertex set of  $K_{22,22,22,55}$  be  $\{0, 1, \dots, 120\}$  partitioned into  $\{3j + i : j = 0, 1, \dots, 21\}$ ,  $i = 0, 1, 2$ , and  $\{66, 67, \dots, 120\}$ . The decompositions consist of

(98, 14, 32, 103, 53, 66, 46, 112, 63, 52, 83,  
44, 18, 97, 35, 39, 43, 101, 62, 102, 96, 0)<sub>L1</sub>,  
(32, 0, 72, 16, 59, 61, 50, 18, 14, 109, 49,  
106, 37, 9, 116, 6, 21, 34, 66, 95, 52, 2)<sub>L1</sub>,  
(107, 26, 37, 42, 86, 17, 85, 36, 81, 7, 3,  
11, 40, 32, 4, 1, 15, 90, 21, 104, 109, 43)<sub>L1</sub>,  
(60, 29, 19, 42, 91, 41, 38, 64, 95, 59, 112,  
58, 83, 55, 63, 30, 85, 46, 98, 50, 24, 104)<sub>L1</sub>,  
(30, 83, 104, 29, 119, 42, 19, 23, 116, 37, 100,  
63, 85, 2, 33, 25, 84, 4, 60, 20, 16, 118)<sub>L1</sub>,  
(105, 10, 4, 92, 45, 104, 9, 111, 67, 19, 93,  
32, 3, 103, 39, 56, 41, 109, 49, 72, 79, 12)<sub>L1</sub>,  
(25, 38, 92, 27, 85, 22, 5, 32, 41, 80, 102,  
58, 99, 56, 75, 9, 50, 33, 108, 78, 24, 16)<sub>L1</sub>,  
(107, 58, 43, 24, 112, 30, 95, 62, 25, 53, 59,  
116, 37, 115, 109, 46, 72, 13, 3, 48, 74, 23)<sub>L2</sub>,  
(91, 50, 39, 88, 9, 34, 44, 49, 56, 18, 84,  
1, 63, 104, 71, 59, 0, 20, 13, 112, 32, 40)<sub>L2</sub>,  
(26, 39, 100, 75, 27, 50, 20, 10, 41, 105, 30,  
93, 21, 13, 94, 92, 115, 0, 46, 57, 78, 35)<sub>L2</sub>,  
(96, 49, 11, 21, 101, 10, 72, 31, 85, 60, 52,  
23, 9, 26, 65, 98, 84, 30, 61, 20, 111, 24)<sub>L2</sub>,  
(71, 61, 21, 90, 1, 73, 58, 5, 19, 2, 100,  
12, 98, 42, 84, 6, 110, 22, 113, 25, 44, 97)<sub>L2</sub>,  
(2, 57, 106, 71, 8, 43, 18, 96, 17, 87, 86,  
24, 32, 33, 22, 53, 110, 6, 77, 10, 59, 114)<sub>L2</sub>,  
(2, 16, 28, 102, 13, 83, 38, 12, 17, 114, 109,  
36, 104, 25, 73, 27, 87, 6, 78, 56, 22, 68)<sub>L2</sub>,  
(30, 58, 7, 78, 48, 9, 112, 113, 44, 17, 39,  
62, 94, 105, 19, 85, 4, 12, 51, 87, 16, 99)<sub>L3</sub>,  
(57, 92, 2, 38, 41, 45, 86, 43, 46, 78, 51,  
68, 20, 1, 16, 36, 100, 87, 80, 21, 61, 11)<sub>L3</sub>,

(96, 43, 39, 29, 74, 91, 44, 17, 31, 16, 93,  
 113, 60, 24, 12, 6, 87, 46, 80, 41, 106, 14) $_{L3}$ ,  
 (1, 84, 32, 101, 43, 40, 107, 93, 14, 29, 58,  
 36, 111, 103, 81, 2, 54, 48, 33, 47, 74, 5) $_{L3}$ ,  
 (100, 63, 8, 64, 72, 49, 46, 115, 9, 86, 85,  
 61, 109, 30, 7, 17, 44, 104, 114, 90, 37, 18) $_{L3}$ ,  
 (84, 11, 48, 59, 63, 115, 96, 38, 10, 43, 58,  
 18, 110, 108, 98, 67, 6, 42, 44, 7, 17, 77) $_{L3}$ ,  
 (53, 3, 1, 114, 56, 69, 72, 5, 37, 15, 26,  
 6, 76, 86, 88, 92, 38, 55, 43, 61, 62, 36) $_{L3}$ ,  
 (110, 8, 3, 9, 27, 81, 41, 40, 69, 25, 48,  
 71, 62, 89, 32, 44, 61, 31, 105, 78, 51, 18) $_{L4}$ ,  
 (88, 11, 13, 59, 28, 80, 93, 65, 103, 19, 37,  
 116, 7, 62, 86, 27, 102, 83, 31, 6, 17, 63) $_{L4}$ ,  
 (105, 63, 34, 37, 5, 81, 111, 27, 112, 41, 33,  
 113, 18, 70, 59, 42, 71, 4, 80, 49, 89, 12) $_{L4}$ ,  
 (17, 4, 102, 107, 48, 21, 35, 20, 11, 28, 49,  
 40, 82, 80, 66, 0, 43, 8, 55, 5, 114, 104) $_{L4}$ ,  
 (86, 57, 8, 32, 58, 117, 106, 52, 82, 27, 63,  
 87, 4, 75, 67, 18, 35, 28, 50, 15, 119, 99) $_{L4}$ ,  
 (58, 60, 87, 80, 74, 92, 48, 53, 5, 34, 3,  
 12, 73, 83, 98, 47, 32, 4, 28, 54, 104, 9) $_{L4}$ ,  
 (41, 24, 45, 108, 85, 84, 10, 4, 63, 2, 17,  
 81, 65, 105, 89, 9, 1, 61, 19, 35, 88, 56) $_{L4}$ ,  
 (110, 23, 21, 52, 95, 16, 2, 113, 15, 53, 81,  
 4, 1, 106, 64, 105, 101, 57, 35, 79, 25, 13) $_{L5}$ ,  
 (57, 37, 80, 50, 51, 75, 17, 53, 119, 110, 18,  
 33, 42, 40, 114, 71, 102, 9, 20, 44, 45, 4) $_{L5}$ ,  
 (34, 115, 93, 29, 48, 0, 65, 53, 97, 71, 20,  
 63, 108, 14, 1, 98, 45, 24, 103, 105, 50, 49) $_{L5}$ ,  
 (89, 48, 13, 27, 55, 72, 17, 56, 112, 76, 11,  
 57, 64, 53, 105, 113, 91, 88, 7, 26, 63, 45) $_{L5}$ ,  
 (75, 1, 48, 5, 79, 26, 62, 87, 33, 19, 106,  
 66, 17, 116, 14, 41, 9, 36, 49, 40, 107, 74) $_{L5}$ ,  
 (101, 52, 53, 33, 97, 20, 12, 7, 25, 11, 89,  
 94, 17, 74, 43, 24, 108, 5, 30, 98, 34, 92) $_{L5}$ ,  
 (30, 13, 99, 46, 75, 112, 25, 29, 27, 84, 58,  
 48, 113, 28, 67, 44, 42, 92, 10, 4, 68, 21) $_{L5}$ ,  
 (41, 6, 111, 18, 81, 112, 22, 51, 20, 58, 26,  
 108, 71, 45, 25, 24, 9, 61, 13, 107, 53, 97) $_{L6}$ ,

(8, 97, 68, 106, 59, 35, 21, 61, 9, 12, 89,  
 83, 78, 80, 52, 2, 50, 0, 19, 29, 100, 105)<sub>L6</sub>,  
 (69, 35, 37, 57, 117, 45, 8, 44, 16, 94, 34,  
 54, 78, 21, 36, 118, 52, 26, 110, 40, 71, 1)<sub>L6</sub>,  
 (91, 38, 37, 23, 49, 108, 72, 47, 6, 74, 21,  
 0, 90, 64, 95, 8, 52, 30, 86, 62, 15, 13)<sub>L6</sub>,  
 (91, 8, 20, 13, 55, 88, 114, 64, 27, 118, 4,  
 32, 90, 66, 53, 77, 6, 41, 2, 36, 103, 54)<sub>L6</sub>,  
 (5, 60, 103, 12, 10, 69, 63, 64, 84, 58, 50,  
 112, 41, 8, 116, 29, 114, 28, 48, 102, 6, 110)<sub>L6</sub>,  
 (18, 107, 40, 25, 16, 0, 109, 99, 70, 60, 100,  
 4, 26, 8, 6, 111, 22, 80, 89, 28, 77, 23)<sub>L6</sub>,  
 (22, 62, 48, 44, 89, 91, 2, 35, 45, 71, 12,  
 19, 92, 98, 47, 77, 31, 51, 17, 32, 39, 49)<sub>L7</sub>,  
 (40, 51, 113, 95, 84, 61, 57, 9, 0, 13, 78,  
 23, 92, 107, 70, 45, 26, 65, 22, 64, 15, 75)<sub>L7</sub>,  
 (41, 70, 91, 106, 40, 20, 5, 25, 59, 61, 89,  
 79, 78, 52, 60, 58, 11, 64, 6, 109, 13, 103)<sub>L7</sub>,  
 (24, 72, 26, 104, 5, 15, 115, 78, 13, 56, 43,  
 53, 62, 112, 107, 83, 99, 88, 49, 8, 19, 45)<sub>L7</sub>,  
 (69, 62, 30, 53, 6, 73, 89, 52, 114, 117, 5,  
 55, 110, 0, 48, 42, 12, 24, 50, 88, 7, 101)<sub>L7</sub>,  
 (79, 10, 44, 20, 96, 86, 19, 34, 3, 92, 6,  
 62, 101, 65, 16, 106, 37, 28, 115, 51, 95, 57)<sub>L7</sub>,  
 (100, 1, 9, 35, 17, 73, 96, 44, 60, 105, 27,  
 24, 85, 10, 58, 112, 3, 21, 106, 47, 102, 52)<sub>L7</sub>,  
 (11, 63, 37, 79, 85, 87, 39, 3, 14, 43, 8,  
 80, 74, 19, 110, 31, 26, 96, 40, 38, 18, 48)<sub>L8</sub>,  
 (36, 104, 38, 67, 15, 9, 109, 113, 52, 0, 119,  
 59, 24, 92, 74, 48, 11, 14, 61, 16, 75, 76)<sub>L8</sub>,  
 (100, 8, 19, 57, 33, 24, 78, 76, 112, 95, 94,  
 25, 18, 34, 59, 26, 96, 54, 55, 13, 9, 118)<sub>L8</sub>,  
 (119, 0, 25, 50, 47, 1, 108, 115, 37, 109, 96,  
 8, 49, 112, 31, 17, 114, 6, 59, 2, 61, 107)<sub>L8</sub>,  
 (17, 48, 70, 60, 118, 22, 36, 15, 32, 101, 33,  
 64, 41, 76, 7, 28, 102, 20, 62, 77, 11, 103)<sub>L8</sub>,  
 (94, 1, 46, 21, 15, 35, 93, 3, 26, 115, 107,  
 52, 66, 27, 12, 40, 54, 108, 68, 34, 88, 5)<sub>L8</sub>,  
 (17, 25, 86, 64, 116, 120, 37, 29, 45, 67, 56,  
 27, 73, 23, 50, 31, 19, 75, 101, 60, 77, 32)<sub>L8</sub>,

(0, 91, 58, 112, 53, 18, 114, 84, 7, 8, 113,  
 12, 55, 74, 117, 2, 110, 27, 38, 56, 54, 70)<sub>L9</sub>,  
 (115, 10, 19, 61, 82, 116, 9, 3, 63, 95, 7,  
 35, 8, 101, 41, 24, 78, 106, 25, 17, 33, 65)<sub>L9</sub>,  
 (6, 69, 83, 13, 24, 1, 35, 48, 77, 75, 118,  
 43, 106, 3, 45, 11, 36, 5, 49, 84, 14, 103)<sub>L9</sub>,  
 (108, 7, 0, 46, 86, 118, 34, 55, 44, 114, 45,  
 38, 65, 72, 49, 83, 81, 42, 6, 29, 9, 43)<sub>L9</sub>,  
 (93, 11, 53, 47, 28, 63, 107, 85, 80, 73, 90,  
 36, 32, 9, 48, 29, 61, 116, 104, 13, 99, 57)<sub>L9</sub>,  
 (98, 6, 5, 16, 87, 70, 40, 15, 11, 105, 19,  
 54, 58, 55, 50, 117, 66, 111, 44, 14, 7, 28)<sub>L9</sub>,  
 (5, 4, 109, 58, 112, 97, 3, 8, 11, 63, 1,  
 91, 69, 12, 119, 45, 23, 32, 37, 117, 55, 70)<sub>L9</sub>,  
 (68, 54, 1, 39, 64, 78, 8, 5, 113, 88, 17,  
 30, 102, 50, 43, 75, 55, 40, 114, 84, 15, 3)<sub>L10</sub>,  
 (95, 56, 16, 51, 112, 109, 47, 38, 17, 58, 54,  
 83, 91, 60, 82, 35, 50, 53, 97, 7, 39, 40)<sub>L10</sub>,  
 (7, 81, 82, 106, 39, 27, 56, 35, 25, 48, 96,  
 93, 37, 109, 40, 61, 107, 66, 15, 41, 42, 50)<sub>L10</sub>,  
 (62, 10, 9, 116, 70, 105, 25, 40, 16, 30, 8,  
 92, 99, 74, 94, 15, 22, 12, 21, 32, 3, 81)<sub>L10</sub>,  
 (28, 5, 103, 107, 108, 46, 49, 57, 26, 35, 76,  
 106, 90, 105, 33, 52, 25, 30, 17, 44, 100, 68)<sub>L10</sub>,  
 (3, 92, 113, 109, 9, 40, 14, 56, 59, 10, 87,  
 74, 105, 66, 15, 52, 51, 12, 47, 53, 85, 84)<sub>L10</sub>,  
 (29, 57, 120, 61, 71, 47, 18, 28, 32, 36, 114,  
 40, 56, 31, 19, 60, 7, 92, 75, 54, 118, 20)<sub>L10</sub>,  
 (64, 105, 90, 59, 44, 43, 51, 52, 16, 108, 93,  
 92, 20, 53, 1, 42, 85, 99, 112, 45, 56, 40)<sub>L11</sub>,  
 (9, 74, 118, 109, 17, 22, 12, 29, 31, 44, 39,  
 27, 115, 76, 35, 86, 42, 65, 19, 36, 62, 67)<sub>L11</sub>,  
 (45, 10, 113, 108, 67, 114, 16, 32, 35, 24, 19,  
 38, 48, 97, 0, 105, 71, 41, 17, 69, 54, 58)<sub>L11</sub>,  
 (12, 115, 22, 86, 28, 39, 71, 97, 8, 35, 43,  
 16, 7, 101, 65, 27, 114, 30, 74, 70, 25, 31)<sub>L11</sub>,  
 (11, 80, 42, 46, 52, 30, 88, 4, 73, 53, 108,  
 43, 45, 87, 50, 6, 101, 9, 72, 91, 58, 32)<sub>L11</sub>,  
 (61, 53, 45, 112, 82, 74, 26, 103, 27, 48, 46,  
 41, 2, 99, 93, 94, 4, 60, 51, 32, 30, 101)<sub>L11</sub>,  
 (51, 85, 95, 31, 17, 29, 43, 64, 118, 116, 119,  
 19, 53, 38, 9, 56, 115, 96, 117, 16, 48, 21)<sub>L11</sub>,

(65, 42, 99, 12, 111, 116, 35, 22, 29, 78, 56,  
 50, 48, 54, 6, 114, 43, 73, 14, 110, 7, 0)<sub>L12</sub>,  
 (106, 61, 63, 46, 107, 118, 34, 17, 62, 105, 25,  
 38, 109, 39, 91, 102, 78, 33, 14, 43, 65, 31)<sub>L12</sub>,  
 (109, 25, 12, 43, 69, 112, 7, 110, 8, 63, 36,  
 17, 22, 114, 66, 48, 78, 62, 59, 60, 56, 58)<sub>L12</sub>,  
 (67, 53, 7, 65, 98, 99, 35, 101, 1, 88, 16,  
 40, 42, 55, 72, 107, 20, 64, 6, 76, 36, 87)<sub>L12</sub>,  
 (40, 109, 113, 81, 19, 48, 8, 16, 0, 3, 91,  
 104, 93, 79, 55, 64, 32, 39, 51, 54, 117, 90)<sub>L12</sub>,  
 (111, 32, 52, 59, 101, 0, 63, 29, 34, 105, 108,  
 94, 90, 56, 17, 11, 54, 45, 70, 61, 100, 47)<sub>L12</sub>,  
 (88, 19, 12, 6, 27, 33, 82, 92, 75, 5, 62,  
 1, 37, 72, 70, 14, 115, 8, 24, 42, 4, 55)<sub>L12</sub>,  
 (119, 35, 8, 64, 93, 77, 1, 45, 51, 27, 59,  
 42, 81, 85, 104, 103, 2, 44, 25, 46, 21, 82)<sub>L13</sub>,  
 (43, 98, 80, 66, 20, 49, 12, 44, 4, 13, 103,  
 73, 101, 60, 53, 46, 30, 19, 75, 110, 2, 10)<sub>L13</sub>,  
 (106, 9, 5, 3, 102, 112, 30, 75, 22, 99, 11,  
 38, 21, 89, 58, 118, 41, 17, 2, 72, 13, 49)<sub>L13</sub>,  
 (56, 30, 84, 92, 104, 79, 24, 3, 43, 17, 52,  
 36, 41, 14, 4, 77, 16, 94, 98, 65, 44, 9)<sub>L13</sub>,  
 (18, 112, 97, 95, 54, 25, 41, 30, 5, 40, 103,  
 11, 68, 6, 109, 86, 1, 32, 108, 22, 51, 36)<sub>L13</sub>,  
 (72, 28, 42, 54, 84, 30, 37, 81, 13, 110, 14,  
 119, 50, 111, 38, 32, 80, 39, 55, 100, 91, 33)<sub>L13</sub>,  
 (92, 14, 34, 22, 37, 12, 45, 110, 93, 70, 55,  
 96, 58, 20, 33, 1, 66, 30, 106, 65, 35, 21)<sub>L13</sub>,  
 (107, 23, 6, 12, 55, 118, 98, 95, 96, 61, 26,  
 33, 8, 60, 78, 25, 10, 84, 113, 45, 39, 65)<sub>L14</sub>,  
 (22, 73, 83, 91, 8, 59, 24, 4, 0, 49, 110,  
 90, 70, 19, 107, 35, 28, 33, 21, 58, 32, 2)<sub>L14</sub>,  
 (65, 118, 112, 68, 38, 57, 39, 31, 63, 7, 80,  
 84, 72, 61, 106, 49, 46, 59, 21, 2, 8, 99)<sub>L14</sub>,  
 (103, 32, 39, 5, 30, 111, 34, 79, 102, 45, 19,  
 40, 64, 18, 109, 93, 80, 89, 24, 2, 46, 9)<sub>L14</sub>,  
 (21, 34, 90, 84, 41, 26, 10, 64, 45, 11, 69,  
 105, 81, 62, 55, 0, 53, 37, 38, 67, 82, 6)<sub>L14</sub>,  
 (12, 22, 82, 99, 87, 103, 28, 35, 19, 23, 45,  
 4, 67, 81, 40, 77, 56, 24, 53, 33, 101, 66)<sub>L14</sub>,  
 (55, 26, 89, 79, 66, 96, 54, 3, 20, 59, 32,  
 16, 76, 110, 43, 80, 0, 18, 45, 56, 23, 75)<sub>L14</sub>,

(8, 101, 36, 93, 56, 17, 13, 111, 4, 23, 39,  
 85, 11, 115, 49, 89, 46, 97, 5, 110, 30, 40)<sub>L15</sub>,  
 (12, 100, 102, 103, 8, 52, 15, 44, 46, 35, 86,  
 82, 37, 26, 43, 0, 14, 95, 78, 54, 19, 2)<sub>L15</sub>,  
 (13, 83, 94, 38, 36, 42, 1, 37, 72, 116, 77,  
 81, 12, 35, 23, 47, 7, 29, 30, 103, 67, 0)<sub>L15</sub>,  
 (36, 28, 105, 102, 68, 72, 17, 54, 39, 27, 22,  
 65, 87, 49, 92, 85, 73, 33, 106, 35, 47, 4)<sub>L15</sub>,  
 (71, 13, 27, 58, 78, 86, 55, 83, 5, 59, 40,  
 30, 19, 68, 70, 42, 113, 15, 1, 54, 50, 69)<sub>L15</sub>,  
 (58, 82, 81, 36, 20, 55, 22, 28, 79, 110, 99,  
 5, 45, 19, 60, 50, 84, 32, 89, 46, 15, 119)<sub>L15</sub>,  
 (51, 17, 91, 35, 84, 57, 9, 45, 1, 99, 52,  
 80, 76, 21, 63, 5, 12, 11, 22, 59, 75, 69)<sub>L15</sub>,  
 (38, 33, 103, 37, 102, 52, 23, 35, 27, 67, 79,  
 70, 42, 25, 41, 11, 58, 26, 54, 10, 111, 117)<sub>L16</sub>,  
 (90, 54, 15, 40, 66, 119, 58, 102, 38, 94, 61,  
 11, 51, 111, 64, 100, 57, 105, 53, 2, 25, 42)<sub>L16</sub>,  
 (6, 23, 113, 116, 77, 21, 1, 51, 42, 12, 75,  
 115, 101, 50, 53, 34, 26, 2, 64, 110, 93, 8)<sub>L16</sub>,  
 (61, 48, 66, 2, 55, 26, 60, 35, 39, 111, 58,  
 110, 95, 64, 45, 47, 24, 49, 101, 113, 89, 22)<sub>L16</sub>,  
 (72, 14, 12, 62, 61, 43, 79, 108, 82, 13, 21,  
 23, 29, 63, 66, 22, 28, 110, 50, 4, 36, 102)<sub>L16</sub>,  
 (61, 80, 76, 72, 45, 62, 26, 22, 25, 54, 104,  
 84, 114, 63, 53, 5, 41, 27, 94, 119, 88, 57)<sub>L16</sub>,  
 (27, 1, 17, 103, 98, 73, 60, 113, 21, 25, 2,  
 4, 37, 77, 88, 72, 116, 99, 16, 39, 3, 8)<sub>L16</sub>,  
 (36, 100, 10, 74, 2, 33, 113, 95, 22, 41, 8,  
 117, 60, 106, 64, 92, 59, 58, 110, 34, 50, 21)<sub>L17</sub>,  
 (6, 101, 96, 81, 61, 56, 14, 50, 48, 21, 36,  
 71, 63, 66, 49, 79, 26, 102, 53, 9, 38, 57)<sub>L17</sub>,  
 (53, 43, 76, 58, 99, 18, 47, 17, 57, 119, 34,  
 91, 117, 66, 52, 65, 15, 41, 54, 25, 55, 79)<sub>L17</sub>,  
 (118, 56, 51, 34, 19, 115, 17, 113, 107, 119, 5,  
 49, 16, 32, 42, 73, 3, 108, 98, 58, 33, 35)<sub>L17</sub>,  
 (76, 58, 40, 22, 97, 78, 48, 95, 39, 117, 35,  
 11, 64, 4, 42, 70, 119, 20, 69, 8, 77, 54)<sub>L17</sub>,  
 (106, 19, 41, 18, 78, 90, 57, 103, 29, 52, 49,  
 10, 15, 105, 117, 30, 119, 37, 58, 54, 44, 80)<sub>L17</sub>,  
 (43, 68, 5, 85, 0, 34, 79, 16, 8, 15, 50,  
 117, 75, 51, 20, 97, 49, 63, 17, 94, 93, 45)<sub>L17</sub>,



(72, 43, 7, 3, 73, 53, 57, 87, 41, 55, 75,  
 84, 62, 61, 45, 60, 8, 11, 92, 119, 47, 6)<sub>L18</sub>,  
 (33, 106, 89, 102, 65, 27, 4, 38, 51, 15, 64,  
 37, 94, 101, 25, 119, 56, 103, 7, 60, 24, 68)<sub>L18</sub>,  
 (40, 8, 75, 41, 115, 85, 28, 25, 37, 6, 50,  
 57, 96, 93, 33, 78, 19, 83, 18, 26, 38, 104)<sub>L18</sub>,  
 (6, 115, 52, 80, 48, 34, 8, 71, 55, 25, 75,  
 73, 41, 14, 67, 33, 64, 49, 36, 23, 81, 39)<sub>L18</sub>,  
 (28, 113, 117, 82, 47, 19, 18, 12, 15, 29, 0,  
 112, 84, 79, 38, 92, 36, 51, 52, 86, 50, 4)<sub>L18</sub>,  
 (111, 49, 28, 8, 77, 104, 17, 112, 9, 93, 7,  
 33, 65, 20, 75, 44, 69, 23, 66, 64, 13, 41)<sub>L18</sub>,  
 (9, 70, 81, 52, 11, 27, 4, 63, 91, 94, 85,  
 83, 44, 115, 58, 22, 17, 6, 14, 111, 68, 12)<sub>L18</sub>,  
 (33, 113, 31, 91, 42, 48, 86, 114, 19, 22, 14,  
 119, 37, 46, 67, 110, 11, 25, 12, 23, 15, 84)<sub>L19</sub>,  
 (83, 42, 59, 2, 34, 35, 69, 25, 86, 40, 101,  
 75, 24, 110, 28, 65, 10, 36, 7, 108, 13, 23)<sub>L19</sub>,  
 (37, 79, 95, 33, 28, 47, 8, 59, 73, 82, 83,  
 96, 52, 106, 12, 23, 19, 26, 108, 104, 34, 56)<sub>L19</sub>,  
 (49, 96, 83, 2, 56, 11, 27, 40, 109, 117, 103,  
 112, 24, 21, 7, 65, 43, 47, 72, 115, 39, 44)<sub>L19</sub>,  
 (111, 3, 7, 51, 26, 109, 117, 82, 31, 58, 6,  
 18, 1, 24, 14, 85, 86, 107, 33, 27, 37, 11)<sub>L19</sub>,  
 (98, 47, 19, 26, 33, 55, 92, 73, 78, 22, 27,  
 102, 16, 60, 25, 90, 14, 0, 110, 66, 53, 42)<sub>L19</sub>,  
 (44, 3, 9, 115, 85, 74, 11, 75, 23, 30, 53,  
 0, 57, 8, 24, 71, 89, 40, 94, 97, 18, 35)<sub>L19</sub>

and

(65, 15, 19, 51, 77, 98, 12, 118, 37, 110, 57,  
 5, 28, 63, 79, 49, 82, 81, 56, 3, 109, 1)<sub>L20</sub>,  
 (56, 49, 82, 88, 47, 111, 0, 59, 6, 46, 34,  
 14, 69, 67, 23, 68, 95, 21, 38, 4, 51, 22)<sub>L20</sub>,  
 (0, 72, 117, 95, 7, 10, 50, 25, 38, 45, 59,  
 90, 92, 42, 85, 13, 101, 62, 61, 18, 84, 36)<sub>L20</sub>,  
 (3, 19, 93, 70, 68, 83, 5, 60, 61, 4, 1,  
 18, 79, 89, 51, 106, 67, 62, 57, 55, 65, 49)<sub>L20</sub>,  
 (77, 41, 58, 51, 89, 99, 53, 98, 28, 81, 32,  
 54, 33, 17, 90, 11, 110, 71, 5, 26, 83, 36)<sub>L20</sub>,  
 (64, 0, 113, 42, 31, 81, 35, 63, 71, 75, 61,  
 40, 111, 53, 19, 14, 96, 69, 39, 44, 77, 2)<sub>L20</sub>,

(44, 58, 87, 42, 95, 90, 63, 41, 8, 37, 36,  
 32, 49, 30, 119, 66, 84, 78, 60, 19, 56, 22)<sub>L20</sub>  
 under the action of the mapping  $x \mapsto x + 3 \pmod{66}$  for  $x < 66$ ,  $x \mapsto$   
 $66 + (x - 66 + 5 \pmod{55})$  for  $x \geq 66$ .

Let the vertex set of  $K_{22,22,22,22,22,55}$  be  $\{0, 1, \dots, 186\}$  partitioned  
 into  $\{6j + i : j = 0, 1, \dots, 21\}$ ,  $i = 0, 1, \dots, 5$ , and  $\{132, 133, \dots, 186\}$ . The  
 decompositions consist of

(181, 64, 78, 15, 138, 94, 150, 126, 132, 16, 28,  
 5, 106, 79, 130, 128, 84, 51, 48, 179, 116, 118)<sub>L1</sub>,  
 (178, 27, 80, 79, 82, 73, 87, 48, 10, 125, 135,  
 101, 140, 91, 85, 43, 146, 120, 15, 47, 182, 8)<sub>L1</sub>,  
 (92, 173, 165, 91, 20, 111, 80, 73, 14, 33, 22,  
 150, 35, 178, 102, 87, 4, 26, 176, 32, 118, 89)<sub>L1</sub>,  
 (142, 59, 111, 104, 113, 173, 134, 123, 14, 117, 105,  
 77, 166, 72, 73, 146, 67, 110, 29, 114, 108, 115)<sub>L1</sub>,  
 (180, 75, 41, 155, 29, 156, 28, 72, 21, 83, 167,  
 76, 138, 88, 32, 50, 127, 9, 149, 63, 38, 154)<sub>L1</sub>,  
 (73, 26, 155, 48, 145, 46, 65, 6, 140, 27, 121,  
 113, 99, 98, 67, 152, 86, 167, 7, 54, 179, 10)<sub>L1</sub>,  
 (18, 31, 38, 50, 143, 65, 35, 10, 109, 169, 162,  
 51, 25, 30, 33, 8, 156, 103, 3, 6, 12, 81)<sub>L1</sub>,  
 (87, 151, 109, 127, 113, 139, 149, 117, 157, 108, 106,  
 7, 181, 67, 23, 123, 51, 13, 147, 148, 12, 27)<sub>L1</sub>,  
 (1, 184, 52, 20, 161, 26, 171, 91, 57, 46, 2,  
 88, 148, 113, 107, 35, 72, 157, 102, 164, 77, 112)<sub>L1</sub>,  
 (60, 150, 2, 67, 92, 22, 132, 153, 20, 134, 47,  
 97, 76, 3, 105, 21, 24, 8, 173, 122, 79, 11)<sub>L1</sub>,  
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 16, 159, 52, 162, 68, 122, 103, 126, 54, 73, 119)<sub>L2</sub>,  
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 17, 94, 79, 60, 181, 133, 5, 32, 20, 126, 185)<sub>L2</sub>,  
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 164, 7, 8, 156, 183, 43, 81, 80, 21, 63, 76)<sub>L2</sub>,  
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 76, 107, 63, 144, 88, 102, 97, 78, 71, 27, 152)<sub>L2</sub>,  
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 143, 41, 1, 70, 126, 164, 21, 150, 50, 123, 132)<sub>L2</sub>,  
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 56, 179, 34, 158, 138, 66, 109, 33, 67, 24, 8)<sub>L2</sub>,

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 132, 58, 116, 83, 128, 30, 166, 59, 167, 180, 37)<sub>L2</sub>,  
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 56, 85, 35, 31, 16, 92, 176, 117, 67, 97, 137)<sub>L2</sub>,  
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 180, 66, 53, 136, 78, 144, 86, 57, 83, 175, 7)<sub>L3</sub>,  
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 72, 56, 47, 120, 172, 11, 150, 20, 147, 131, 79)<sub>L3</sub>,  
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 44, 34, 69, 4, 90, 27, 46, 178, 97, 71, 35)<sub>L3</sub>,  
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 6, 69, 45, 23, 128, 118, 73, 48, 182, 84, 158)<sub>L3</sub>,  
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 178, 18, 34, 75, 59, 76, 125, 67, 120, 106, 96)<sub>L3</sub>,  
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 15, 42, 4, 23, 168, 20, 176, 63, 114, 55, 7)<sub>L3</sub>,  
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 90, 94, 101, 127, 35, 80, 74, 102, 145, 105, 168)<sub>L3</sub>,  
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 53, 74, 90, 180, 174, 124, 186, 77, 127, 63, 20)<sub>L3</sub>,  
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 75, 11, 102, 58, 47, 168, 5, 144, 9, 70, 1)<sub>L4</sub>,  
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 85, 90, 183, 94, 163, 128, 14, 51, 66, 17, 142)<sub>L4</sub>,  
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 46, 135, 90, 10, 148, 33, 137, 126, 163, 95, 58)<sub>L4</sub>,  
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 39, 116, 19, 23, 154, 36, 62, 3, 161, 94, 37)<sub>L4</sub>,  
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 34, 85, 90, 6, 156, 80, 10, 121, 117, 120, 134)<sub>L4</sub>,  
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 45, 110, 107, 7, 85, 145, 49, 152, 0, 71, 81)<sub>L4</sub>,  
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 31, 29, 147, 161, 167, 120, 60, 45, 105, 20, 44)<sub>L4</sub>,  
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 153, 120, 179, 71, 130, 77, 101, 68, 145, 66, 109)<sub>L5</sub>,  
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 57, 97, 14, 153, 142, 66, 19, 28, 23, 3, 12)<sub>L5</sub>,  
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 178, 164, 116, 119, 4, 11, 88, 81, 96, 179, 174)<sub>L5</sub>,  
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 83, 99, 74, 98, 51, 174, 1, 105, 176, 54, 38)<sub>L5</sub>,  
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 172, 42, 61, 158, 72, 112, 41, 33, 16, 26, 28)<sub>L5</sub>,  
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 132, 117, 154, 100, 7, 56, 85, 66, 114, 152, 175)<sub>L5</sub>,  
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 98, 99, 14, 85, 58, 160, 52, 116, 126, 23, 15)<sub>L5</sub>,  
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 73, 120, 110, 29, 104, 171, 184, 111, 93, 37, 12)<sub>L6</sub>,  
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 152, 97, 119, 65, 19, 28, 135, 111, 118, 75, 46)<sub>L6</sub>,  
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 115, 16, 14, 73, 45, 157, 145, 7, 65, 0, 64)<sub>L6</sub>,  
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 33, 171, 22, 102, 17, 94, 126, 15, 176, 59, 148)<sub>L6</sub>,  
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 156, 17, 125, 145, 96, 48, 54, 99, 172, 45, 70)<sub>L6</sub>,  
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 37, 33, 166, 110, 24, 144, 95, 14, 66, 31, 186)<sub>L6</sub>,  
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 98, 142, 139, 118, 177, 42, 58, 44, 62, 167, 87)<sub>L7</sub>,  
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 126, 137, 54, 168, 165, 66, 53, 51, 94, 56, 30)<sub>L7</sub>,  
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 94, 184, 61, 12, 47, 115, 112, 156, 29, 26, 120)<sub>L7</sub>,  
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 35, 181, 64, 60, 111, 25, 104, 160, 103, 147, 78)<sub>L7</sub>,  
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 87, 103, 154, 113, 83, 125, 107, 150, 123, 7, 34)<sub>L8</sub>,  
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 145, 175, 61, 174, 86, 121, 59, 89, 88, 8, 147)<sub>L9</sub>,  
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 36, 8, 43, 23, 122, 1, 71, 174, 45, 176, 114)<sub>L9</sub>,  
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 103, 39, 163, 30, 175, 73, 158, 118, 69, 167, 123)<sub>L10</sub>,  
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 87, 98, 76, 148, 16, 111, 35, 166, 72, 32, 31)<sub>L10</sub>,  
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 83, 44, 51, 0, 99, 67, 176, 8, 178, 31, 72)<sub>L12</sub>,  
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 51, 12, 174, 67, 159, 85, 164, 119, 39, 116, 60)<sub>L12</sub>,  
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 65, 130, 136, 135, 75, 180, 96, 44, 10, 36, 178)<sub>L15</sub>,  
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 138, 63, 173, 25, 98, 40, 115, 26, 175, 67, 96)<sub>L15</sub>,  
 (136, 40, 103, 36, 91, 92, 112, 131, 124, 57, 30,  
 69, 85, 170, 176, 165, 110, 46, 8, 19, 7, 113)<sub>L15</sub>,  
 (102, 141, 119, 73, 37, 92, 99, 51, 172, 163, 154,  
 160, 161, 97, 80, 7, 55, 4, 82, 59, 120, 56)<sub>L16</sub>,  
 (42, 182, 5, 136, 56, 125, 70, 172, 91, 53, 8,  
 102, 68, 112, 169, 146, 167, 7, 128, 76, 82, 24)<sub>L16</sub>,  
 (185, 66, 114, 117, 52, 146, 128, 88, 157, 36, 83,  
 11, 1, 33, 40, 135, 165, 99, 107, 131, 42, 126)<sub>L16</sub>,  
 (79, 178, 140, 18, 40, 19, 104, 42, 148, 162, 21,  
 75, 11, 33, 83, 157, 120, 74, 133, 173, 119, 69)<sub>L16</sub>,  
 (110, 91, 166, 141, 5, 154, 37, 9, 48, 119, 74,  
 55, 156, 85, 175, 41, 94, 124, 107, 25, 135, 30)<sub>L16</sub>,  
 (23, 164, 61, 74, 84, 86, 15, 101, 45, 16, 146,  
 30, 60, 159, 33, 70, 92, 56, 131, 18, 19, 165)<sub>L16</sub>,  
 (172, 47, 34, 22, 74, 180, 24, 38, 72, 14, 10,  
 7, 158, 119, 60, 69, 174, 25, 76, 132, 35, 96)<sub>L16</sub>,  
 (165, 96, 73, 55, 47, 63, 133, 24, 12, 44, 151,  
 153, 180, 131, 53, 48, 127, 9, 130, 22, 169, 107)<sub>L16</sub>,  
 (151, 70, 107, 112, 133, 93, 100, 37, 105, 152, 162,  
 115, 84, 159, 113, 108, 24, 7, 13, 144, 16, 92)<sub>L16</sub>,  
 (42, 4, 67, 113, 134, 108, 9, 92, 56, 158, 117,  
 64, 99, 138, 116, 65, 0, 11, 97, 90, 31, 144)<sub>L16</sub>,  
 (59, 171, 180, 36, 10, 94, 19, 129, 31, 86, 110,  
 43, 82, 18, 120, 57, 8, 152, 83, 138, 4, 6)<sub>L17</sub>,  
 (70, 174, 14, 73, 69, 74, 152, 136, 66, 33, 114,  
 84, 85, 23, 149, 128, 143, 125, 6, 64, 176, 101)<sub>L17</sub>,  
 (135, 86, 123, 126, 52, 11, 48, 139, 33, 149, 58,  
 50, 71, 103, 76, 156, 94, 121, 142, 57, 145, 49)<sub>L17</sub>,  
 (109, 93, 64, 47, 152, 115, 101, 21, 114, 143, 0,  
 41, 54, 141, 12, 11, 121, 46, 133, 98, 66, 44)<sub>L17</sub>,

(185, 43, 110, 52, 4, 175, 154, 84, 23, 42, 118,  
 74, 155, 26, 91, 129, 164, 115, 27, 54, 77, 176)<sub>L17</sub>,  
 (46, 92, 158, 133, 159, 83, 9, 58, 55, 40, 172,  
 48, 80, 105, 167, 5, 107, 155, 100, 143, 79, 162)<sub>L17</sub>,  
 (68, 149, 27, 148, 89, 47, 2, 65, 105, 51, 22,  
 100, 52, 5, 95, 90, 158, 0, 80, 82, 1, 176)<sub>L17</sub>,  
 (143, 55, 29, 76, 88, 126, 60, 153, 17, 166, 81,  
 146, 102, 10, 67, 35, 73, 58, 70, 59, 137, 99)<sub>L17</sub>,  
 (180, 12, 105, 98, 88, 103, 144, 130, 139, 34, 179,  
 40, 162, 84, 99, 101, 147, 117, 125, 14, 104, 108)<sub>L17</sub>,  
 (98, 51, 83, 161, 16, 125, 150, 12, 39, 108, 140,  
 1, 92, 54, 32, 24, 11, 135, 29, 141, 63, 162)<sub>L17</sub>,  
 (72, 39, 110, 151, 31, 19, 118, 160, 35, 90, 58,  
 113, 5, 66, 109, 149, 82, 180, 52, 102, 103, 142)<sub>L18</sub>,  
 (177, 112, 1, 82, 138, 125, 114, 143, 0, 171, 25,  
 8, 22, 31, 127, 96, 77, 60, 63, 106, 14, 179)<sub>L18</sub>,  
 (165, 23, 119, 25, 52, 22, 63, 130, 114, 180, 24,  
 110, 45, 5, 154, 127, 36, 78, 123, 10, 32, 83)<sub>L18</sub>,  
 (0, 144, 40, 55, 86, 130, 131, 152, 65, 104, 68,  
 13, 60, 125, 43, 163, 182, 76, 139, 117, 22, 47)<sub>L18</sub>,  
 (159, 11, 90, 93, 133, 164, 55, 168, 65, 42, 19,  
 121, 15, 28, 167, 162, 54, 138, 71, 44, 27, 79)<sub>L18</sub>,  
 (174, 101, 48, 54, 118, 147, 160, 105, 171, 1, 81,  
 120, 22, 10, 60, 182, 65, 97, 162, 87, 83, 78)<sub>L18</sub>,  
 (7, 166, 71, 164, 47, 88, 108, 38, 61, 92, 173,  
 8, 29, 3, 27, 24, 175, 5, 176, 144, 89, 78)<sub>L18</sub>,  
 (54, 20, 136, 71, 45, 159, 2, 116, 145, 48, 50,  
 99, 100, 6, 69, 157, 59, 161, 94, 91, 89, 166)<sub>L18</sub>,  
 (176, 44, 80, 106, 70, 109, 23, 54, 175, 155, 52,  
 118, 185, 133, 104, 71, 47, 95, 114, 75, 110, 163)<sub>L18</sub>,  
 (152, 66, 50, 81, 59, 178, 60, 146, 25, 135, 83,  
 102, 7, 31, 5, 43, 156, 63, 64, 78, 183, 15)<sub>L18</sub>,  
 (122, 94, 85, 139, 51, 26, 14, 63, 37, 27, 52,  
 95, 145, 7, 56, 178, 167, 30, 65, 148, 68, 61)<sub>L19</sub>,  
 (91, 183, 178, 159, 73, 119, 24, 14, 123, 88, 20,  
 148, 163, 90, 44, 42, 92, 21, 58, 12, 32, 157)<sub>L19</sub>,  
 (19, 133, 48, 126, 123, 114, 10, 149, 107, 44, 182,  
 82, 9, 131, 30, 53, 162, 171, 83, 37, 140, 7)<sub>L19</sub>,  
 (124, 181, 156, 96, 4, 86, 51, 63, 137, 68, 81,  
 78, 36, 154, 56, 1, 17, 14, 134, 18, 161, 37)<sub>L19</sub>,  
 (48, 109, 40, 52, 168, 157, 1, 66, 129, 74, 118,  
 81, 139, 77, 141, 97, 170, 107, 86, 2, 88, 57)<sub>L19</sub>,

(15, 172, 127, 18, 6, 29, 50, 165, 98, 97, 102,  
 110, 10, 25, 83, 157, 69, 153, 115, 117, 74, 34)<sub>L19</sub>,  
 (133, 100, 104, 115, 47, 38, 140, 181, 16, 117, 147,  
 39, 44, 85, 174, 8, 161, 96, 175, 26, 43, 29)<sub>L19</sub>,  
 (107, 171, 118, 68, 84, 75, 131, 156, 100, 43, 76,  
 158, 79, 2, 9, 35, 51, 66, 80, 136, 38, 145)<sub>L19</sub>,  
 (139, 55, 101, 67, 10, 53, 26, 19, 62, 155, 98,  
 43, 144, 170, 87, 127, 108, 175, 66, 54, 69, 169)<sub>L19</sub>,  
 (8, 134, 146, 155, 55, 29, 117, 54, 40, 96, 118,  
 67, 49, 157, 57, 177, 78, 33, 106, 126, 7, 129)<sub>L19</sub>

and

(83, 157, 97, 60, 20, 37, 179, 78, 47, 105, 4,  
 164, 26, 175, 49, 14, 87, 99, 150, 156, 9, 82)<sub>L20</sub>,  
 (114, 105, 89, 130, 65, 160, 26, 106, 75, 47, 49,  
 112, 81, 68, 25, 182, 86, 148, 128, 18, 70, 67)<sub>L20</sub>,  
 (174, 48, 80, 51, 172, 28, 36, 175, 61, 163, 113,  
 8, 67, 122, 146, 23, 165, 136, 138, 64, 6, 32)<sub>L20</sub>,  
 (160, 100, 85, 84, 6, 49, 70, 64, 92, 154, 22,  
 166, 111, 167, 81, 66, 89, 36, 80, 79, 134, 141)<sub>L20</sub>,  
 (79, 23, 124, 40, 160, 73, 125, 184, 17, 164, 129,  
 77, 68, 72, 76, 118, 115, 86, 138, 151, 89, 83)<sub>L20</sub>,  
 (78, 133, 56, 157, 68, 100, 36, 118, 33, 73, 139,  
 129, 105, 185, 83, 167, 47, 122, 175, 91, 21, 96)<sub>L20</sub>,  
 (41, 118, 57, 34, 17, 176, 167, 16, 87, 131, 20,  
 50, 183, 137, 59, 54, 93, 83, 36, 7, 10, 58)<sub>L20</sub>,  
 (67, 168, 0, 80, 15, 31, 173, 2, 177, 131, 88,  
 169, 60, 78, 99, 56, 180, 51, 135, 98, 82, 12)<sub>L20</sub>,  
 (154, 94, 11, 128, 132, 177, 64, 16, 41, 33, 92,  
 119, 27, 108, 109, 30, 176, 85, 84, 48, 15, 76)<sub>L20</sub>,  
 (154, 131, 96, 85, 168, 93, 37, 181, 51, 183, 136,  
 14, 76, 99, 80, 25, 21, 48, 116, 151, 88, 133)<sub>L20</sub>

under the action of the mapping  $x \mapsto x + 3 \pmod{132}$  for  $x < 132$ ,  $x \mapsto 132 + (x - 132 + 5 \pmod{55})$  for  $x \geq 132$ .  $\square$

Theorem 1.4 follows from Lemmas 5.1, 5.2 and Proposition 1.7.

## 6 Goldberg's snark #3

Goldberg's snark #3 is represented by the ordered 24-tuple of its vertices  $(1, 2, \dots, 24)_{\text{GS3}}$ . Its edge set, as supplied with the MATHEMATICA system, [24], is  $\{\{9, 10\}, \{8, 9\}, \{9, 13\}, \{13, 16\}, \{15, 16\}, \{16, 17\}, \{10, 17\}, \{8, 15\}, \{5, 7\}, \{1, 3\}, \{1, 2\}, \{2, 5\}, \{2, 4\}, \{4, 6\}, \{6, 7\}, \{3, 6\}, \{18, 20\}, \{22, 24\}, \{19, 22\}, \{18, 19\}, \{19, 21\}, \{21, 23\}, \{20, 23\}, \{23, 24\},$

$\{7, 15\}, \{12, 13\}, \{1, 10\}, \{3, 22\}, \{4, 11\}, \{17, 24\}, \{14, 21\}, \{5, 20\}, \{8, 18\}, \{11, 14\}, \{12, 14\}, \{11, 12\}$ .

**Lemma 6.1** *There exist Goldberg's snark #3 designs of order 64, 73, 136 and 145.*

**Proof.** Let the vertex set of  $K_{64}$  be  $Z_{63} \cup \{\infty\}$ . The decomposition consists of

$(\infty, 11, 58, 48, 37, 22, 13, 33, 26, 21, 6, 54,$   
 $60, 45, 10, 0, 8, 52, 19, 2, 5, 14, 59, 57)_{\text{GS3}},$   
 $(62, 24, 28, 58, 51, 3, 49, 8, 23, 1, 57, 45,$   
 $41, 26, 54, 2, 22, 30, 60, 37, 42, 13, 31, 0)_{\text{GS3}},$   
 $(58, 15, 35, 14, 5, 56, 26, 60, 25, 50, 2, 30,$   
 $29, 34, 1, 46, 62, 32, 16, 22, 23, 28, 53, 0)_{\text{GS3}},$   
 $(40, 36, 52, 8, 16, 44, 34, 10, 2, 12, 19, 50,$   
 $49, 23, 11, 4, 55, 20, 31, 17, 56, 35, 29, 32)_{\text{GS3}}$

under the action of the mapping  $\infty \mapsto \infty, x \mapsto x + 3 \pmod{63}$  for the first two graphs, and  $x \mapsto x + 9 \pmod{63}$  for the last two.

Let the vertex set of  $K_{73}$  be  $Z_{73}$ . The decomposition consists of  
 $(42, 3, 33, 22, 53, 28, 7, 11, 14, 25, 0, 1,$   
 $16, 8, 21, 34, 5, 36, 71, 12, 38, 45, 54, 9)_{\text{GS3}}$

under the action of the mapping  $x \mapsto x + 1 \pmod{73}$ .

Let the vertex set of  $K_{136}$  be  $Z_{135} \cup \{\infty\}$ . The decomposition consists of

$(\infty, 76, 105, 41, 117, 33, 51, 62, 85, 17, 19, 110,$   
 $116, 13, 104, 81, 39, 25, 57, 18, 79, 111, 119, 58)_{\text{GS3}},$   
 $(114, 46, 53, 101, 70, 84, 41, 45, 16, 35, 26, 117,$   
 $2, 62, 97, 86, 98, 12, 67, 80, 65, 118, 7, 55)_{\text{GS3}},$   
 $(31, 134, 119, 20, 75, 72, 86, 29, 16, 32, 60, 131,$   
 $23, 3, 62, 14, 40, 68, 85, 125, 108, 67, 43, 28)_{\text{GS3}},$   
 $(37, 67, 94, 117, 33, 61, 20, 3, 103, 17, 9, 19,$   
 $53, 96, 18, 49, 76, 112, 5, 108, 122, 30, 2, 75)_{\text{GS3}},$   
 $(14, 119, 124, 29, 78, 99, 83, 53, 7, 15, 126, 68,$   
 $96, 105, 102, 92, 28, 107, 38, 75, 43, 84, 63, 25)_{\text{GS3}},$   
 $(97, 99, 108, 60, 115, 103, 54, 4, 13, 88, 16, 114,$   
 $67, 30, 78, 69, 52, 123, 84, 72, 121, 10, 100, 57)_{\text{GS3}},$   
 $(88, 0, 67, 121, 9, 31, 48, 55, 97, 7, 79, 51,$   
 $37, 100, 99, 82, 93, 13, 73, 120, 64, 78, 118, 102)_{\text{GS3}}$

under the action of the mapping  $\infty \mapsto \infty, x \mapsto x + 3 \pmod{135}$  for the first five graphs, and  $x \mapsto x + 9 \pmod{135}$  for the last two.

Let the vertex set of  $K_{145}$  be  $Z_{145}$ . The decomposition consists of  
 $(2, 106, 141, 113, 58, 22, 38, 62, 85, 74, 0, 1,$   
 $3, 4, 5, 13, 8, 6, 15, 18, 28, 30, 45, 59)_{\text{GS3}},$

(0, 18, 19, 39, 40, 64, 2, 1, 29, 59, 3, 34,  
71, 73, 51, 4, 103, 82, 24, 5, 126, 84, 74, 13)<sub>GS3</sub>  
under the action of the mapping  $x \mapsto x + 1 \pmod{145}$ .  $\square$

**Lemma 6.2** *There exist decompositions into Goldberg's snark #3 of the complete multipartite graphs  $K_{12,12,12}$ ,  $K_{24,24,15}$ ,  $K_{72,72,63}$ ,  $K_{24,24,24,24}$  and  $K_{24,24,24,21}$ .*

**Proof.** Let the vertex set of  $K_{12,12,12}$  be  $Z_{36}$  partitioned according to residue classes modulo 3. The decomposition consists of

(11, 30, 12, 13, 28, 14, 24, 18, 19, 6, 2, 7,  
35, 27, 32, 22, 29, 10, 15, 3, 1, 8, 23, 25)<sub>GS3</sub>

under the action of the mapping  $x \mapsto x + 3 \pmod{36}$ .

Let the vertex set of  $K_{24,24,15}$  be  $\{0, 1, \dots, 62\}$  partitioned into  $\{2j + i : j = 0, 1, \dots, 23\}$ ,  $i = 0, 1$ , and  $\{48, 49, \dots, 62\}$ . The decompositions consist of

(10, 27, 13, 12, 61, 56, 32, 18, 17, 55, 39, 50,  
40, 8, 29, 54, 47, 58, 19, 25, 60, 22, 42, 52)<sub>GS3</sub>,  
(16, 52, 37, 1, 12, 20, 23, 48, 27, 58, 54, 14,  
49, 9, 30, 35, 36, 0, 43, 19, 61, 10, 46, 11)<sub>GS3</sub>,  
(37, 32, 2, 23, 17, 61, 42, 24, 57, 30, 14, 50,  
33, 29, 13, 20, 7, 54, 10, 8, 58, 39, 9, 52)<sub>GS3</sub>

under the action of the mapping  $x \mapsto x + 4 \pmod{48}$  for  $x < 48$ ,  $x \mapsto 48 + (x - 48 + 5 \pmod{15})$  for  $x \geq 48$ .

Let the vertex set of  $K_{72,72,63}$  be  $\{0, 1, \dots, 206\}$  partitioned into  $\{2j + i : j = 0, 1, \dots, 71\}$ ,  $i = 0, 1$ , and  $\{144, 145, \dots, 206\}$ . The decompositions consist of

(186, 47, 13, 140, 171, 169, 62, 82, 188, 43, 99, 120,  
95, 158, 174, 78, 147, 39, 130, 8, 65, 164, 165, 139)<sub>GS3</sub>,  
(189, 48, 14, 141, 172, 170, 63, 83, 191, 44, 100, 121,  
96, 159, 175, 79, 148, 40, 131, 9, 66, 165, 166, 140)<sub>GS3</sub>,  
(192, 49, 15, 142, 173, 171, 64, 84, 194, 45, 101, 122,  
97, 160, 176, 80, 149, 41, 132, 10, 67, 166, 167, 141)<sub>GS3</sub>,  
(195, 50, 16, 143, 174, 172, 65, 85, 197, 46, 102, 123,  
98, 161, 177, 81, 150, 42, 133, 11, 68, 167, 168, 142)<sub>GS3</sub>,  
(150, 83, 32, 84, 164, 47, 86, 191, 13, 6, 174, 52,  
203, 55, 1, 90, 153, 115, 18, 142, 202, 113, 3, 136)<sub>GS3</sub>,  
(151, 84, 33, 85, 165, 48, 87, 194, 14, 7, 175, 53,  
206, 56, 2, 91, 154, 116, 19, 143, 205, 114, 4, 137)<sub>GS3</sub>,  
(152, 85, 34, 86, 166, 49, 88, 197, 15, 8, 176, 54,  
182, 57, 3, 92, 155, 117, 20, 0, 181, 115, 5, 138)<sub>GS3</sub>,  
(153, 86, 35, 87, 167, 50, 89, 200, 16, 9, 177, 55,  
185, 58, 4, 93, 156, 118, 21, 1, 184, 116, 6, 139)<sub>GS3</sub>,

$(32, 141, 3, 171, 84, 22, 13, 143, 148, 61, 12, 89,$   
 $118, 180, 82, 47, 28, 158, 128, 9, 109, 190, 184, 63)_{\text{GS3}},$   
 $(102, 153, 89, 87, 112, 186, 39, 64, 199, 121, 78, 3,$   
 $60, 196, 192, 127, 50, 193, 22, 125, 105, 183, 38, 5)_{\text{GS3}},$   
 $(192, 46, 48, 180, 79, 57, 140, 116, 201, 38, 3, 187,$   
 $6, 12, 183, 35, 193, 47, 2, 92, 196, 15, 137, 102)_{\text{GS3}}$   
 under the action of the mapping  $x \mapsto x + 4 \pmod{144}$  for  $x < 144$ ,  $x \mapsto$   
 $144 + (x + 4 \pmod{36})$  for  $144 \leq x < 180$ ,  $x \mapsto 180 + (x - 180 + 12 \pmod{27})$   
 for  $x \geq 180$ .

Let the vertex set of  $K_{24,24,24,24}$  be  $Z_{96}$  partitioned according to residue  
 classes modulo 4. The decomposition consists of  
 $(33, 62, 24, 92, 51, 85, 36, 44, 47, 84, 91, 78,$   
 $72, 37, 18, 41, 2, 6, 80, 49, 3, 34, 82, 7)_{\text{GS3}}$   
 under the action of the mapping  $x \mapsto x + 1 \pmod{96}$ .

Let the vertex set of  $K_{24,24,24,21}$  be  $\{0, 1, \dots, 92\}$  partitioned into  $\{2j+i :$   
 $j = 0, 1, \dots, 23\}$ ,  $i = 0, 1, 2$ , and  $\{72, 73, \dots, 92\}$ . The decompositions  
 consist of

$(13, 72, 44, 57, 65, 81, 48, 34, 8, 45, 83, 41,$   
 $10, 46, 53, 18, 80, 59, 42, 88, 47, 19, 15, 20)_{\text{GS3}},$   
 $(88, 68, 45, 60, 30, 58, 20, 63, 2, 54, 53, 61,$   
 $91, 6, 85, 5, 84, 29, 48, 1, 75, 19, 51, 23)_{\text{GS3}},$   
 $(38, 75, 37, 41, 61, 57, 85, 28, 59, 48, 67, 3,$   
 $90, 5, 47, 54, 35, 17, 31, 60, 15, 73, 32, 6)_{\text{GS3}},$   
 $(49, 42, 35, 28, 11, 91, 48, 27, 78, 24, 90, 5,$   
 $66, 7, 61, 38, 82, 81, 30, 46, 87, 72, 68, 3)_{\text{GS3}},$   
 $(44, 40, 4, 17, 63, 87, 11, 2, 76, 24, 21, 75,$   
 $28, 43, 70, 14, 85, 52, 36, 84, 72, 86, 26, 66)_{\text{GS3}}$   
 under the action of the mapping  $x \mapsto x + 4 \pmod{72}$  for  $x < 72$ ,  $x \mapsto$   
 $72 + (x - 72 + 7 \pmod{21})$  for  $x \geq 72$ .  $\square$

Theorem 1.5 follows from Lemmas 6.1, 6.2 and Proposition 1.8.

## 7 The two Celmins–Swart snarks and the two 26-vertex Blanuša snarks

The Celmins–Swart snarks and the 26-vertex Blanuša snarks are each repre-  
 sented by the ordered 26-tuple of its vertices:  $(1, 2, \dots, 26)_{\text{CS1}}$  for Celmins–  
 Swart snark #1,  $(1, 2, \dots, 26)_{\text{CS2}}$  for Celmins–Swart snark #2,  $(1, 2, \dots,$   
 $26)_{\text{B21}}$  for the  $(2, 1)$ -Blanuša snark and  $(1, 2, \dots, 26)_{\text{B22}}$  for the  $(2, 2)$ -  
 Blanuša snark. The edge sets, as supplied with the MATHEMATICA system,  
 [24], are respectively

CS1:  $\{\{1, 2\}, \{1, 4\}, \{1, 10\}, \{2, 6\}, \{2, 8\}, \{3, 4\}, \{3, 5\}, \{3, 8\}, \{4, 7\}, \{5, 6\}, \{5, 12\}, \{6, 7\}, \{7, 9\}, \{8, 11\}, \{9, 10\}, \{9, 14\}, \{10, 18\}, \{11, 15\}, \{11, 16\}, \{12, 13\}, \{12, 16\}, \{13, 14\}, \{13, 15\}, \{14, 17\}, \{15, 22\}, \{16, 19\}, \{17, 18\}, \{17, 21\}, \{18, 26\}, \{19, 23\}, \{19, 25\}, \{20, 21\}, \{20, 22\}, \{20, 25\}, \{21, 24\}, \{22, 23\}, \{23, 24\}, \{24, 26\}, \{25, 26\}\},$

CS2:  $\{\{1, 6\}, \{1, 2\}, \{3, 4\}, \{3, 7\}, \{2, 4\}, \{3, 5\}, \{5, 6\}, \{1, 7\}, \{7, 20\}, \{8, 9\}, \{8, 17\}, \{9, 10\}, \{4, 10\}, \{10, 14\}, \{6, 11\}, \{11, 12\}, \{12, 13\}, \{12, 16\}, \{9, 13\}, \{13, 18\}, \{14, 15\}, \{14, 19\}, \{2, 15\}, \{15, 26\}, \{8, 16\}, \{16, 21\}, \{11, 17\}, \{17, 18\}, \{18, 19\}, \{19, 24\}, \{20, 25\}, \{21, 22\}, \{5, 22\}, \{22, 23\}, \{20, 23\}, \{23, 24\}, \{24, 26\}, \{21, 25\}, \{25, 26\}\},$

B21:  $\{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 6\}, \{3, 5\}, \{4, 6\}, \{4, 8\}, \{5, 7\}, \{5, 8\}, \{6, 7\}, \{7, 9\}, \{8, 10\}, \{9, 11\}, \{9, 12\}, \{10, 11\}, \{10, 14\}, \{11, 13\}, \{12, 14\}, \{12, 16\}, \{13, 15\}, \{13, 16\}, \{14, 15\}, \{15, 17\}, \{16, 18\}, \{17, 19\}, \{17, 20\}, \{18, 19\}, \{18, 22\}, \{19, 21\}, \{20, 22\}, \{20, 24\}, \{21, 23\}, \{21, 24\}, \{22, 23\}, \{23, 25\}, \{24, 26\}, \{1, 25\}, \{2, 26\}, \{25, 26\}\}$  and

B22:  $\{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 6\}, \{4, 8\}, \{5, 7\}, \{5, 8\}, \{6, 7\}, \{7, 9\}, \{8, 10\}, \{9, 11\}, \{9, 12\}, \{10, 11\}, \{10, 14\}, \{11, 13\}, \{12, 14\}, \{12, 16\}, \{13, 15\}, \{13, 16\}, \{14, 15\}, \{15, 17\}, \{16, 18\}, \{17, 19\}, \{17, 20\}, \{18, 19\}, \{18, 22\}, \{19, 21\}, \{20, 22\}, \{20, 24\}, \{21, 23\}, \{21, 24\}, \{22, 23\}, \{23, 1\}, \{24, 2\}, \{3, 25\}, \{4, 26\}, \{5, 25\}, \{6, 26\}, \{25, 26\}\}.$

**Lemma 7.1** *There exist designs of order 40 and 79 for each of the two Celmins–Swart snarks and the two 26-vertex Blanuša snarks.*

**Proof.** Let the vertex set of  $K_{40}$  be  $Z_{40}$ . The decompositions consist of

$(15, 20, 1, 11, 30, 37, 10, 2, 7, 16, 22, 6, 0,$   
 $27, 23, 31, 36, 4, 17, 19, 38, 5, 35, 12, 33, 32)_{CS1},$   
 $(36, 30, 29, 15, 9, 7, 23, 25, 10, 32, 21, 12, 1,$   
 $20, 16, 28, 5, 34, 35, 18, 26, 27, 19, 2, 33, 6)_{CS1},$   
 $(30, 35, 37, 22, 7, 13, 21, 0, 27, 33, 19, 5, 17,$   
 $23, 4, 16, 20, 25, 9, 11, 38, 14, 26, 29, 18, 8)_{CS1},$   
 $(11, 26, 29, 13, 6, 32, 18, 28, 23, 12, 15, 24, 14, 16,$   
 $3, 0, 8, 21, 36, 38, 17, 31, 10, 39, 2, 9)_{CS1},$   
 $(1, 34, 24, 28, 9, 31, 19, 8, 10, 32, 18, 4, 3,$   
 $12, 36, 11, 6, 2, 25, 22, 30, 15, 26, 13, 33, 27)_{CS2},$   
 $(20, 37, 36, 4, 15, 23, 8, 30, 12, 14, 26, 18, 33,$   
 $7, 24, 3, 38, 25, 27, 5, 32, 0, 10, 31, 13, 17)_{CS2},$   
 $(32, 38, 9, 33, 0, 31, 18, 2, 19, 36, 37, 22, 11,$   
 $26, 29, 1, 13, 35, 5, 23, 12, 17, 21, 20, 6, 15)_{CS2},$   
 $(27, 0, 36, 7, 23, 39, 22, 24, 29, 9, 11, 2, 20,$   
 $21, 19, 18, 25, 35, 31, 32, 38, 5, 16, 30, 14, 13)_{CS2},$

(12, 6, 32, 36, 25, 26, 18, 31, 5, 22, 34, 16, 37,  
 14, 33, 11, 17, 24, 28, 7, 30, 29, 21, 27, 3, 10)<sub>B21</sub>,  
 (22, 19, 35, 16, 24, 15, 21, 6, 36, 31, 0, 2, 18,  
 28, 3, 20, 38, 17, 5, 9, 4, 14, 13, 23, 37, 7)<sub>B21</sub>,  
 (0, 2, 16, 19, 28, 18, 33, 20, 6, 7, 35, 23, 12,  
 26, 38, 25, 22, 37, 3, 4, 1, 14, 10, 11, 9, 8)<sub>B21</sub>,  
 (22, 10, 15, 23, 33, 8, 29, 16, 34, 24, 7, 17, 18,  
 9, 27, 37, 1, 39, 35, 20, 3, 31, 5, 13, 28, 0)<sub>B21</sub>

and

(1, 5, 0, 19, 17, 4, 26, 32, 37, 3, 29, 31, 13,  
 9, 33, 10, 20, 18, 28, 35, 15, 30, 36, 27, 34, 21)<sub>B22</sub>,  
 (31, 1, 6, 29, 0, 33, 19, 32, 23, 13, 24, 8, 34,  
 30, 9, 14, 7, 5, 11, 4, 2, 35, 18, 37, 20, 22)<sub>B22</sub>,  
 (36, 18, 14, 17, 34, 12, 35, 27, 10, 28, 32, 15, 22,  
 37, 38, 3, 30, 23, 31, 19, 20, 6, 8, 16, 1, 7)<sub>B22</sub>,  
 (29, 28, 4, 16, 0, 35, 33, 7, 5, 31, 17, 15, 8,  
 22, 34, 32, 38, 9, 13, 1, 39, 12, 10, 2, 14, 11)<sub>B22</sub>

under the action of the mapping  $x \mapsto x + 8 \pmod{40}$ .

Let the vertex set of  $K_{79}$  be  $Z_{79}$ . The decompositions consist of

(62, 53, 77, 59, 5, 19, 74, 6, 51, 24, 17, 3, 56,  
 7, 57, 13, 1, 29, 0, 9, 55, 21, 42, 71, 31, 12)<sub>CS1</sub>,  
 (17, 24, 76, 77, 12, 22, 68, 57, 20, 74, 0, 40, 38,  
 7, 53, 2, 9, 3, 23, 6, 45, 59, 25, 46, 72, 42)<sub>CS2</sub>,  
 (69, 10, 42, 12, 14, 46, 52, 0, 67, 71, 38, 36, 3,  
 15, 40, 6, 1, 2, 11, 8, 66, 68, 7, 61, 70, 72)<sub>B21</sub>

and

(42, 73, 38, 32, 69, 72, 2, 48, 43, 37, 0, 41, 34,  
 18, 66, 1, 4, 6, 12, 29, 25, 58, 28, 47, 10, 17)<sub>B22</sub>

under the action of the mapping  $x \mapsto x + 1 \pmod{79}$ . □

**Lemma 7.2** *There exist decompositions of  $K_{39,39,39}$  and  $K_{13,13,13,13}$  into each of the two Celmins–Swart snarks and the two 26-vertex Blanuša snarks.*

**Proof.** Let the vertex set of  $K_{39,39,39}$  be  $Z_{117}$  partitioned according to residue classes modulo 3. The decompositions consist of

(64, 44, 53, 72, 111, 85, 5, 10, 78, 92, 32, 26, 51,  
 68, 28, 19, 0, 31, 3, 8, 46, 90, 91, 86, 73, 84)<sub>CS1</sub>,  
 (27, 94, 49, 54, 66, 101, 98, 50, 84, 107, 102, 5, 19,  
 79, 87, 3, 40, 8, 21, 0, 47, 10, 41, 37, 25, 116)<sub>CS2</sub>,  
 (75, 53, 40, 5, 8, 51, 85, 22, 78, 17, 79, 115, 11,  
 90, 112, 21, 3, 1, 29, 34, 48, 77, 19, 98, 71, 109)<sub>B21</sub>



and

(47, 25, 72, 43, 59, 110, 10, 45, 62, 67, 75, 31, 11,  
87, 4, 12, 9, 1, 44, 82, 90, 27, 106, 2, 100, 33)<sub>B22</sub>

under the action of the mapping  $x \mapsto x + 1 \pmod{117}$ .

Let the vertex set of  $K_{13,13,13,13}$  be  $Z_{52}$  partitioned according to residue classes modulo 4. The decompositions consist of

(15, 34, 11, 0, 26, 21, 39, 4, 1, 12, 35, 3, 32,  
46, 9, 41, 47, 25, 31, 14, 49, 28, 18, 24, 48, 6)<sub>CS1</sub>,  
(4, 31, 49, 26, 28, 21, 23, 32, 6, 5, 34, 13, 16,  
11, 7, 43, 18, 39, 2, 50, 9, 1, 48, 46, 29, 20)<sub>CS1</sub>,

(26, 32, 27, 25, 48, 7, 13, 5, 38, 15, 50, 11, 36,  
18, 41, 20, 16, 3, 8, 39, 35, 34, 4, 9, 46, 19)<sub>CS2</sub>,  
(10, 20, 5, 46, 42, 27, 40, 23, 17, 15, 24, 22, 4,  
16, 41, 29, 1, 18, 49, 11, 30, 33, 6, 0, 25, 7)<sub>CS2</sub>,

(24, 5, 6, 3, 7, 34, 17, 48, 15, 18, 13, 2, 26,  
8, 35, 40, 49, 14, 16, 46, 33, 19, 4, 44, 43, 50)<sub>B21</sub>,  
(18, 7, 37, 45, 19, 30, 41, 24, 50, 49, 15, 35, 12,  
36, 21, 9, 31, 16, 17, 25, 11, 46, 28, 20, 51, 10)<sub>B21</sub>

and

(18, 21, 24, 16, 11, 3, 48, 9, 15, 34, 37, 6, 42,  
27, 33, 23, 28, 13, 26, 19, 12, 32, 49, 22, 10, 29)<sub>B22</sub>,  
(23, 28, 17, 12, 39, 31, 2, 34, 15, 16, 45, 25, 43,  
42, 8, 18, 35, 40, 49, 4, 38, 46, 9, 27, 44, 13)<sub>B22</sub>

under the action of the mapping  $x \mapsto x + 4 \pmod{52}$ . □

Theorem 1.6 (i) follows from Lemmas 7.1, 7.2 and Proposition 1.4.

## 8 The flower snark J7

The flower snark J7 is represented by the ordered 28-tuple of its vertices  $(1, 2, \dots, 28)_{\text{FJ7}}$  and edge set  $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{3, 7\}, \{3, 8\}, \{4, 9\}, \{4, 10\}, \{5, 11\}, \{5, 12\}, \{6, 13\}, \{6, 14\}, \{7, 11\}, \{7, 13\}, \{8, 15\}, \{8, 16\}, \{9, 12\}, \{9, 14\}, \{10, 17\}, \{10, 18\}, \{11, 19\}, \{12, 20\}, \{13, 21\}, \{14, 22\}, \{15, 17\}, \{15, 23\}, \{16, 19\}, \{16, 21\}, \{17, 24\}, \{18, 20\}, \{18, 22\}, \{19, 25\}, \{20, 26\}, \{21, 27\}, \{22, 28\}, \{23, 25\}, \{23, 27\}, \{24, 26\}, \{24, 28\}, \{25, 28\}, \{26, 27\}\}$  as supplied with the MATHEMATICA system, [24].

**Lemma 8.1** *There exist flower snark J7 designs of order 28, 85 and 169.*

**Proof.** Let the vertex set of  $K_{28}$  be  $Z_{27} \cup \{\infty\}$ . The decomposition consists of

$(\infty, 15, 26, 1, 25, 5, 16, 8, 7, 4, 18, 12, 11, 3, 10,$   
 $23, 19, 17, 24, 13, 22, 14, 2, 0, 21, 20, 6, 9)_{\text{FJ7}}$   
 under the action of the mapping  $\infty \mapsto \infty, x \mapsto x + 3 \pmod{27}$ .  
 Let the vertex set of  $K_{85}$  be  $Z_{85}$ . The decomposition consists of  
 $(21, 37, 77, 11, 72, 22, 52, 32, 4, 83, 28, 25, 64, 24,$   
 $58, 0, 1, 2, 5, 3, 6, 10, 27, 56, 38, 39, 73, 19)_{\text{FJ7}}$   
 under the action of the mapping  $x \mapsto x + 1 \pmod{85}$ .  
 Let the vertex set of  $K_{169}$  be  $Z_{169}$ . The decomposition consists of  
 $(117, 136, 116, 167, 165, 145, 149, 121, 130, 92, 78, 4, 90, 142,$   
 $134, 40, 152, 7, 110, 30, 156, 111, 3, 52, 138, 148, 27, 91)_{\text{FJ7}},$   
 $(128, 126, 164, 112, 25, 80, 149, 5, 116, 137, 86, 19, 1, 3,$   
 $12, 16, 26, 9, 121, 36, 68, 83, 69, 75, 99, 151, 24, 161)_{\text{FJ7}}$   
 under the action of the mapping  $x \mapsto x + 1 \pmod{169}$ .  $\square$

**Lemma 8.2** *There exist decompositions of  $K_{42,42,42}$  and  $K_{7,7,7,7}$  into the flower snark  $J7$ .*

**Proof.** Let the vertex set of  $K_{42,42,42}$  be  $Z_{126}$  partitioned according to residue classes modulo 3. The decomposition consists of

$(42, 113, 74, 16, 22, 63, 93, 99, 47, 80, 83, 0, 4, 7,$   
 $1, 2, 3, 6, 10, 5, 15, 89, 123, 37, 17, 117, 100, 105)_{\text{FJ7}}$   
 under the action of the mapping  $x \mapsto x + 1 \pmod{126}$ .

Let the vertex set of  $K_{7,7,7,7}$  be  $Z_{28}$  partitioned according to residue classes modulo 4. The decomposition consists of

$(25, 16, 7, 11, 5, 22, 20, 17, 9, 2, 3, 8, 13, 4,$   
 $10, 23, 27, 1, 24, 18, 0, 26, 12, 6, 15, 19, 14, 21)_{\text{FJ7}}$   
 under the action of the mapping  $x \mapsto x + 4 \pmod{28}$ .  $\square$

**Lemma 8.3** *There exist a flower snark  $J7$  design of order  $n$  if  $n \equiv 28 \pmod{84}$ .*

**Proof.** The case  $n = 28$  follows from Lemma 8.1. There exists a 4-GDD of type  $4^{3t+1}$  for  $t \geq 1$ , [7] (see also [16]). Inflate by a factor of 7, lay a complete graph  $K_{28}$  (from Lemma 8.1) on each of the inflated groups and replace each block by a complete 4-partite graph  $K_{7,7,7,7}$  (from Lemma 8.2) to yield a flower snark  $J7$  design of order  $84t + 28$  for  $t \geq 1$ .  $\square$

Theorem 1.6 (ii) follows from Lemmas 8.1, 8.2 and Proposition 1.1 for  $n \equiv 1 \pmod{84}$ , and Lemma 8.3 for  $n \equiv 28 \pmod{84}$ .

## 9 The double star snark

The double star snark is represented by the ordered 30-tuple of its vertices  $(1, 2, \dots, 30)_{\text{DS}}$  and edge set  $\{\{13, 16\}, \{15, 16\}, \{12, 15\}, \{16, 19\},$

$\{15, 18\}, \{23, 25\}, \{25, 29\}, \{28, 29\}, \{25, 26\}, \{29, 30\}, \{21, 24\}, \{21, 22\}, \{22, 27\}, \{17, 21\}, \{20, 22\}, \{10, 14\}, \{9, 10\}, \{9, 11\}, \{7, 10\}, \{4, 9\}, \{5, 6\}, \{2, 6\}, \{1, 2\}, \{6, 8\}, \{2, 3\}, \{14, 19\}, \{7, 13\}, \{5, 26\}, \{8, 23\}, \{13, 17\}, \{19, 24\}, \{7, 23\}, \{14, 26\}, \{8, 24\}, \{5, 17\}, \{1, 18\}, \{12, 30\}, \{20, 28\}, \{4, 27\}, \{3, 11\}, \{18, 28\}, \{27, 30\}, \{11, 20\}, \{1, 4\}, \{3, 12\}\}$  as supplied with the MATHEMATICA system, [24].

**Lemma 9.1** *There exist double star snark designs of order 46, 91 and 181.*

**Proof.** Let the vertex set of  $K_{46}$  be  $Z_{46}$ . The decomposition consists of  
 $(41, 10, 13, 24, 6, 16, 4, 21, 5, 43, 39, 12, 32, 0, 23,$   
 $28, 19, 27, 8, 2, 40, 17, 38, 15, 36, 22, 33, 37, 9, 11)_{\text{DS}}$   
under the action of the mapping  $x \mapsto x + 2 \pmod{46}$ .

Let the vertex set of  $K_{91}$  be  $Z_{91}$ . The decomposition consists of  
 $(10, 1, 48, 50, 59, 11, 49, 47, 27, 69, 16, 34, 4, 77, 31, 88,$   
 $6, 85, 12, 20, 0, 19, 8, 25, 70, 42, 83, 15, 82, 52)_{\text{DS}}$   
under the action of the mapping  $x \mapsto x + 1 \pmod{91}$ .

Let the vertex set of  $K_{181}$  be  $Z_{181}$ . The decomposition consists of  
 $(158, 77, 118, 129, 105, 50, 98, 83, 70, 25, 133, 143, 14, 162, 117,$   
 $79, 1, 26, 88, 21, 0, 2, 3, 5, 6, 12, 9, 4, 16, 20)_{\text{DS}},$   
 $(0, 8, 22, 16, 1, 29, 2, 5, 34, 54, 57, 52, 36, 3, 4,$   
 $68, 72, 43, 105, 7, 134, 64, 111, 58, 15, 128, 131, 142, 102, 10)_{\text{DS}}$   
under the action of the mapping  $x \mapsto x + 1 \pmod{181}$ .  $\square$

**Lemma 9.2** *There exists a decomposition of  $K_{15,15,15}$  into the double star snark.*

**Proof.** Let the vertex set of  $K_{15,15,15}$  be  $Z_{45}$  partitioned according to residue classes modulo 3. The decomposition consists of  
 $(26, 15, 14, 7, 25, 38, 34, 31, 17, 21, 33, 6, 18, 22, 28,$   
 $32, 5, 36, 27, 37, 19, 39, 24, 2, 43, 20, 41, 29, 12, 40)_{\text{DS}}$   
under the action of the mapping  $x \mapsto x + 3 \pmod{45}$ .  $\square$

Theorem 1.6 (iii) follows from Lemmas 9.1, 9.2 and Proposition 1.3.

## 10 The two 34-vertex Blanuša snarks

The two 34-vertex Blanuša snarks are each represented by the ordered 36-tuple of its vertices,  $(1, 2, \dots, 34)_{\text{B31}}$  for the (3,1)-Blanuša snark and  $(1, 2, \dots, 34)_{\text{B32}}$  for the (3,2)-Blanuša snark. The edge sets, as supplied with the MATHEMATICA system, [24], are respectively

B31:  $\{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 6\}, \{3, 5\}, \{4, 6\}, \{4, 8\}, \{5, 7\}, \{5, 8\}, \{6, 7\}, \{7, 9\}, \{8, 10\}, \{9, 11\}, \{9, 12\}, \{10, 11\}, \{10, 14\}, \{11, 13\}, \{12, 14\},$

$\{12, 16\}, \{13, 15\}, \{13, 16\}, \{14, 15\}, \{15, 17\}, \{16, 18\}, \{17, 19\}, \{17, 20\},$   
 $\{18, 19\}, \{18, 22\}, \{19, 21\}, \{20, 22\}, \{20, 24\}, \{21, 23\}, \{21, 24\}, \{22, 23\},$   
 $\{23, 25\}, \{24, 26\}, \{25, 27\}, \{25, 28\}, \{26, 27\}, \{26, 30\}, \{27, 29\}, \{28, 30\},$   
 $\{28, 32\}, \{29, 31\}, \{29, 32\}, \{30, 31\}, \{31, 33\}, \{32, 34\}, \{33, 1\}, \{34, 2\},$   
 $\{33, 34\}$  and

B32:  $\{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 6\}, \{4, 8\}, \{5, 7\}, \{5, 8\}, \{6, 7\}, \{7, 9\},$   
 $\{8, 10\}, \{9, 11\}, \{9, 12\}, \{10, 11\}, \{10, 14\}, \{11, 13\}, \{12, 14\}, \{12, 16\}, \{13, 15\},$   
 $\{13, 16\}, \{14, 15\}, \{15, 17\}, \{16, 18\}, \{17, 19\}, \{17, 20\}, \{18, 19\}, \{18, 22\},$   
 $\{19, 21\}, \{20, 22\}, \{20, 24\}, \{21, 23\}, \{21, 24\}, \{22, 23\}, \{23, 25\}, \{24, 26\},$   
 $\{25, 27\}, \{25, 28\}, \{26, 27\}, \{26, 30\}, \{27, 29\}, \{28, 30\}, \{28, 32\}, \{29, 31\},$   
 $\{29, 32\}, \{30, 31\}, \{31, 1\}, \{32, 2\}, \{3, 33\}, \{4, 34\}, \{5, 33\}, \{6, 34\}, \{33, 34\}\}.$

**Lemma 10.1** *There exist designs of order 52 and 103 for each of the two 34-vertex Blanuša snarks.*

**Proof.** Let the vertex set of  $K_{52}$  be  $Z_{52}$ . The decomposition consists of

$(43, 6, 27, 36, 38, 41, 17, 12, 39, 30, 22, 10, 7, 28, 25, 50, 32,$   
 $26, 21, 5, 8, 48, 47, 45, 3, 44, 24, 29, 14, 13, 46, 15, 40, 11)_{B31},$   
 $(48, 29, 23, 18, 45, 9, 7, 11, 12, 38, 28, 24, 5, 35, 47, 15, 32,$   
 $39, 26, 36, 33, 19, 0, 34, 14, 50, 25, 17, 27, 13, 21, 46, 4, 42)_{B31}$

and

$(6, 20, 37, 23, 45, 44, 26, 11, 29, 28, 27, 35, 48, 50, 10, 39, 46,$   
 $34, 47, 3, 36, 31, 4, 25, 33, 1, 15, 40, 38, 18, 12, 30, 24, 49)_{B32},$   
 $(11, 40, 3, 47, 44, 38, 45, 28, 8, 14, 41, 0, 32, 12, 35, 39, 13,$   
 $6, 26, 7, 21, 34, 30, 17, 51, 5, 49, 31, 33, 42, 1, 36, 10, 9)_{B32}$

under the action of the mapping  $x \mapsto x + 4 \pmod{52}$ .

Let the vertex set of  $K_{103}$  be  $Z_{103}$ . The decomposition consists of

$(0, 1, 2, 3, 6, 8, 14, 15, 4, 26, 39, 18, 5, 41, 21, 35, 40,$   
 $7, 58, 9, 10, 31, 52, 36, 11, 80, 43, 67, 92, 17, 63, 22, 25, 61)_{B31}$

and

$(0, 1, 2, 3, 4, 5, 10, 11, 19, 21, 6, 7, 17, 35, 51, 34, 8,$   
 $9, 27, 28, 48, 50, 12, 72, 38, 14, 69, 95, 13, 43, 66, 55, 37, 76)_{B32}$

under the action of the mapping  $x \mapsto x + 1 \pmod{103}$ .  $\square$

**Lemma 10.2** *There exists decompositions of  $K_{51,51,51}$  and  $K_{17,17,17,17}$  into each of the two 34-vertex Blanuša snarks.*

**Proof.** Let the vertex set of  $K_{51,51,51}$  be  $Z_{153}$  partitioned according to residue classes modulo 3. The decomposition consists of

$(0, 1, 2, 4, 7, 11, 24, 15, 5, 29, 45, 25, 8, 3, 31, 54, 6,$   
 $10, 41, 38, 75, 81, 13, 124, 60, 47, 148, 19, 69, 108, 14, 119, 97, 39)_{B31}$

and

(0, 1, 2, 4, 3, 6, 13, 17, 5, 28, 45, 21, 7, 47, 69, 41, 8,  
9, 37, 31, 12, 62, 97, 68, 11, 114, 55, 109, 125, 32, 73, 51, 67, 110)<sub>B32</sub>  
under the action of the mapping  $x \mapsto x + 1 \pmod{153}$ .

Let the vertex set of  $K_{17,17,17,17}$  be  $Z_{68}$  partitioned according to residue classes modulo 4. The decomposition consists of

(17, 53, 40, 42, 13, 43, 60, 39, 29, 0, 35, 51, 46, 10, 27, 33, 41,  
16, 15, 8, 38, 62, 23, 37, 61, 19, 54, 14, 49, 64, 26, 48, 36, 50)<sub>B31</sub>,  
(46, 35, 8, 39, 55, 1, 30, 64, 13, 15, 2, 62, 29, 20, 27, 31, 49,  
37, 52, 40, 34, 51, 36, 43, 61, 22, 0, 3, 57, 16, 42, 66, 9, 4)<sub>B31</sub>

and

(64, 34, 59, 29, 35, 56, 42, 26, 7, 13, 40, 10, 1, 12, 22, 44, 61,  
37, 62, 52, 32, 46, 19, 31, 21, 48, 2, 55, 15, 17, 54, 41, 28, 47)<sub>B32</sub>,  
(2, 50, 44, 13, 4, 17, 7, 18, 25, 39, 28, 20, 30, 33, 12, 35, 1,  
8, 46, 47, 63, 61, 10, 24, 11, 67, 22, 42, 49, 57, 51, 0, 21, 43)<sub>B32</sub>

under the action of the mapping  $x \mapsto x + 4 \pmod{68}$ .  $\square$

Theorem 1.6 (iv) follows from Lemmas 10.1, 10.2 and Proposition 1.4.

## 11 Zamfirescu's graph

Zamfirescu's graph is represented by the ordered 36-tuple of its vertices (1, 2, ..., 36)<sub>Z</sub> and edge set  $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{8, 9\}, \{9, 1\}, \{3, 8\}, \{6, 2\}, \{9, 5\}, \{10, 11\}, \{11, 12\}, \{12, 13\}, \{13, 14\}, \{14, 15\}, \{15, 16\}, \{16, 17\}, \{17, 18\}, \{18, 10\}, \{12, 17\}, \{15, 11\}, \{18, 14\}, \{19, 20\}, \{20, 21\}, \{21, 22\}, \{22, 23\}, \{23, 24\}, \{24, 25\}, \{25, 26\}, \{26, 27\}, \{27, 19\}, \{21, 26\}, \{24, 20\}, \{27, 23\}, \{28, 29\}, \{29, 30\}, \{30, 31\}, \{31, 32\}, \{32, 33\}, \{33, 34\}, \{34, 35\}, \{35, 36\}, \{36, 28\}, \{30, 35\}, \{33, 29\}, \{36, 32\}, \{1, 10\}, \{4, 19\}, \{7, 28\}, \{16, 22\}, \{25, 31\}, \{34, 13\}\}$  as supplied with the MATHEMATICA system, [24].

**Lemma 11.1** *There exist Zamfirescu's graph designs of order 109 and 217.*

**Proof.** Let the vertex set of  $K_{109}$  be  $Z_{109}$ . The decomposition consists of

(0, 1, 3, 6, 2, 7, 14, 22, 13, 10, 24, 4, 16, 31, 8, 25, 43, 64,  
28, 52, 5, 50, 9, 78, 11, 39, 66, 45, 80, 17, 83, 51, 15, 76, 46, 104)<sub>Z</sub>

under the action of the mapping  $x \mapsto x + 1 \pmod{109}$ .

Let the vertex set of  $K_{217}$  be  $Z_{217}$ . The decomposition consists of

(193, 8, 55, 59, 182, 146, 209, 22, 98, 184, 20, 149, 178, 74, 122, 79, 40, 102,  
134, 99, 85, 68, 26, 3, 119, 156, 153, 141, 126, 164, 169, 86, 172, 73, 133, 16)<sub>Z</sub>,

(153, 23, 75, 65, 0, 1, 3, 9, 16, 5, 13, 25, 4, 17, 37, 10, 50, 94,

6, 24, 58, 84, 11, 69, 2, 112, 167, 54, 95, 14, 128, 35, 180, 61, 125, 174)<sub>Z</sub>

under the action of the mapping  $x \mapsto x + 1 \pmod{217}$ .  $\square$

**Lemma 11.2** *There exists a decomposition of  $K_{18,18,18}$  into Zamfirescu's graph.*

**Proof.** Let the vertex set of  $K_{18,18,18}$  be  $Z_{54}$  partitioned according to residue classes modulo 3. The decomposition consists of

$$(21, 25, 35, 40, 14, 6, 7, 45, 31, 28, 30, 29, 15, 47, 13, 2, 43, 0, \\ 9, 4, 5, 37, 12, 8, 16, 48, 32, 23, 46, 53, 24, 50, 33, 20, 18, 52)_Z$$

under the action of the mapping  $x \mapsto x + 3 \pmod{54}$ .  $\square$

Theorem 1.6 (v) follows from Lemmas 11.1, 11.2 and Proposition 1.2.

## 12 Goldberg's snark #5

Goldberg's snark #5 is represented by the ordered 40-tuple of its vertices  $(1, 2, \dots, 40)_{\text{GS5}}$ . The edge set is  $\{\{15, 16\}, \{14, 15\}, \{15, 21\}, \{21, 26\}, \{25, 26\}, \{26, 27\}, \{16, 27\}, \{14, 25\}, \{7, 10\}, \{2, 6\}, \{6, 10\}, \{4, 6\}, \{3, 4\}, \{3, 7\}, \{1, 3\}, \{1, 2\}, \{9, 17\}, \{5, 8\}, \{8, 9\}, \{8, 11\}, \{11, 13\}, \{13, 17\}, \{12, 13\}, \{5, 12\}, \{24, 32\}, \{28, 29\}, \{24, 28\}, \{28, 30\}, \{30, 33\}, \{32, 33\}, \{33, 36\}, \{29, 36\}, \{31, 34\}, \{38, 40\}, \{34, 38\}, \{37, 38\}, \{35, 37\}, \{31, 35\}, \{35, 39\}, \{39, 40\}, \{2, 16\}, \{7, 25\}, \{10, 17\}, \{1, 5\}, \{9, 32\}, \{12, 29\}, \{24, 31\}, \{36, 40\}, \{14, 34\}, \{27, 39\}, \{20, 21\}, \{4, 18\}, \{11, 19\}, \{22, 30\}, \{23, 37\}, \{19, 22\}, \{22, 23\}, \{20, 23\}, \{18, 20\}, \{18, 19\}\}$ , as supplied with the MATH-EMATICA system, [24].

**Lemma 12.1** *There exist Goldberg's snark #5 designs of order 40, 121 and 241.*

**Proof.** Let the vertex set of  $K_{40}$  be  $Z_{39} \cup \{\infty\}$ . The decomposition consists of the graphs

$$(\infty, 23, 18, 26, 4, 35, 2, 25, 21, 7, 19, 36, 34, 20, \\ 17, 3, 24, 30, 0, 31, 1, 33, 6, 38, 22, 8, 29, \\ 16, 13, 28, 14, 32, 15, 10, 9, 5, 37, 11, 27, 12)_{\text{GS5}}$$

under the action of the mapping  $\infty \mapsto \infty, x \mapsto x + 3 \pmod{39}$ .

Let the vertex set of  $K_{121}$  be  $Z_{121}$ . The decomposition consists of the graphs

$$(0, 1, 2, 5, 4, 10, 8, 11, 3, 20, 22, 17, 36, \\ 6, 21, 37, 53, 23, 43, 45, 68, 9, 69, 7, 33, 96, \\ 66, 38, 70, 64, 44, 55, 109, 86, 83, 32, 118, 40, 26, 91)_{\text{GS5}}$$

under the action of the mapping  $x \mapsto x + 1 \pmod{121}$ .

Let the vertex set of  $K_{241}$  be  $Z_{241}$ . The decomposition consists of the graphs

(169, 233, 1, 108, 6, 57, 27, 190, 239, 224, 144, 192, 130,  
 39, 187, 216, 202, 74, 134, 142, 84, 158, 178, 32, 136, 127,  
 56, 212, 184, 79, 5, 133, 83, 154, 3, 64, 55, 66, 69, 104)<sub>GS5</sub>,  
 (195, 23, 31, 224, 164, 219, 38, 231, 190, 216, 118, 45, 22,  
 96, 176, 170, 92, 13, 200, 12, 24, 1, 33, 0, 5, 80,  
 41, 15, 98, 91, 87, 86, 232, 198, 172, 214, 147, 88, 125, 230)<sub>GS5</sub>  
 under the action of the mapping  $x \mapsto x + 1 \pmod{241}$ .  $\square$

**Lemma 12.2** *There exist decompositions of the complete multipartite graphs  $K_{60,60,60}$ ,  $K_{20,20,20,20}$  and  $K_{40,40,40,40}$  into Goldberg's snark #5.*

**Proof.** Let the vertex set of  $K_{60,60,60}$  be  $Z_{180}$  partitioned according to residue classes modulo 3. The decomposition consists of the graph

(0, 1, 2, 6, 5, 14, 9, 15, 4, 28, 31, 22, 51,  
 3, 25, 48, 77, 34, 65, 59, 96, 7, 99, 8, 41, 139,  
 83, 49, 101, 93, 58, 78, 173, 122, 113, 55, 10, 63, 16, 131)<sub>GS5</sub>  
 under the action of the mapping  $x \mapsto x + 1 \pmod{180}$ .

Let the vertex set of  $K_{20,20,20,20}$  be  $Z_{80}$  partitioned according to residue classes modulo 4. The decomposition consists of the graph

(31, 48, 56, 18, 1, 5, 29, 50, 41, 71, 43, 47, 38,  
 14, 35, 58, 72, 4, 62, 51, 57, 63, 40, 0, 3, 6,  
 8, 19, 12, 60, 74, 59, 42, 9, 27, 55, 25, 64, 37, 10)<sub>GS5</sub>  
 under the action of the mapping  $x \mapsto x + 2 \pmod{80}$ .

Let the vertex set of  $K_{40,40,40,40}$  be  $Z_{160}$  partitioned according to residue classes modulo 4. The decomposition consists of the graph

(37, 98, 22, 49, 88, 40, 148, 41, 36, 93, 2, 94, 129,  
 78, 15, 8, 6, 59, 21, 109, 150, 139, 24, 0, 1, 4,  
 25, 11, 12, 33, 18, 62, 128, 149, 104, 61, 47, 106, 23, 152)<sub>GS5</sub>  
 under the action of the mapping  $x \mapsto x + 1 \pmod{160}$ .  $\square$

**Lemma 12.3** *There exist a Goldberg's snark #5 design of order  $n$  if  $n \equiv 40 \pmod{120}$ .*

**Proof.** There exist a 4-GDD of type  $2^{3t+1}$  for  $t \geq 2$ , [7] (see also [16]). Inflate by a factor of 20, lay a complete graph  $K_{40}$  from Lemma 12.1 on each of the inflated groups and replace each block by a complete 4-partite graph  $K_{20,20,20,20}$  from Lemma 12.2 to yield a Goldberg's snark #5 design of order  $120t + 40$  for  $t \geq 2$ . The design for the remaining case,  $n = 160$  is constructed from  $K_{40}$  (Lemma 12.1) and the 4-partite graph  $K_{40,40,40,40}$  from Lemma 12.2.  $\square$

Theorem 1.6 (vi) follows from Lemmas 12.1, 12.2 and Proposition 1.1 for  $n \equiv 1 \pmod{120}$ , and Lemma 12.3 for  $n \equiv 40 \pmod{120}$ .

### 13 The Szekeres and Watkins snarks

The Szekeres snark and the Watkins snark are each represented by the ordered 50-tuple of its vertices,  $(1, 2, \dots, 50)_{\text{Sze}}$  for the Szekeres snark and  $(1, 2, \dots, 50)_{\text{Wat}}$  for the Watkins snark. The edge sets, as supplied with the MATHEMATICA system, [24], are respectively

Sze:  $\{\{29, 30\}, \{27, 29\}, \{22, 24\}, \{21, 22\}, \{21, 23\}, \{23, 25\}, \{25, 28\}, \{28, 30\}, \{22, 26\}, \{25, 26\}, \{26, 29\}, \{23, 27\}, \{24, 28\}, \{3, 6\}, \{5, 10\}, \{1, 3\}, \{3, 5\}, \{1, 12\}, \{2, 10\}, \{4, 6\}, \{4, 9\}, \{9, 12\}, \{12, 13\}, \{6, 13\}, \{10, 13\}, \{2, 4\}, \{16, 17\}, \{19, 20\}, \{14, 17\}, \{17, 19\}, \{14, 15\}, \{8, 20\}, \{7, 16\}, \{7, 11\}, \{11, 15\}, \{15, 18\}, \{16, 18\}, \{18, 20\}, \{7, 8\}, \{35, 44\}, \{36, 41\}, \{43, 44\}, \{41, 44\}, \{31, 43\}, \{36, 38\}, \{34, 35\}, \{32, 34\}, \{31, 32\}, \{31, 33\}, \{33, 35\}, \{33, 36\}, \{34, 38\}, \{45, 47\}, \{39, 42\}, \{47, 49\}, \{42, 47\}, \{40, 49\}, \{39, 50\}, \{45, 48\}, \{46, 48\}, \{40, 46\}, \{37, 40\}, \{37, 45\}, \{37, 39\}, \{48, 50\}, \{30, 41\}, \{9, 42\}, \{11, 21\}, \{5, 32\}, \{19, 46\}, \{1, 27\}, \{2, 14\}, \{8, 43\}, \{38, 49\}, \{24, 50\}\}$   
and

Wat:  $\{\{1, 2\}, \{1, 4\}, \{1, 15\}, \{2, 3\}, \{2, 8\}, \{3, 6\}, \{3, 37\}, \{4, 6\}, \{4, 7\}, \{5, 10\}, \{5, 11\}, \{5, 22\}, \{6, 9\}, \{7, 8\}, \{7, 12\}, \{8, 9\}, \{9, 14\}, \{10, 13\}, \{10, 17\}, \{11, 16\}, \{11, 18\}, \{12, 14\}, \{12, 33\}, \{13, 15\}, \{13, 16\}, \{14, 20\}, \{15, 21\}, \{16, 19\}, \{17, 18\}, \{17, 19\}, \{18, 30\}, \{19, 21\}, \{20, 24\}, \{20, 26\}, \{21, 50\}, \{22, 23\}, \{22, 27\}, \{23, 24\}, \{23, 25\}, \{24, 29\}, \{25, 26\}, \{25, 28\}, \{26, 31\}, \{27, 28\}, \{27, 48\}, \{28, 29\}, \{29, 31\}, \{30, 32\}, \{30, 36\}, \{31, 36\}, \{32, 34\}, \{32, 35\}, \{33, 34\}, \{33, 40\}, \{34, 41\}, \{35, 38\}, \{35, 40\}, \{36, 38\}, \{37, 39\}, \{37, 42\}, \{38, 41\}, \{39, 44\}, \{39, 46\}, \{40, 46\}, \{41, 46\}, \{42, 43\}, \{42, 45\}, \{43, 44\}, \{43, 49\}, \{44, 47\}, \{45, 47\}, \{45, 48\}, \{47, 50\}, \{48, 49\}, \{49, 50\}\}$ .

**Lemma 13.1** *There exist Szekeres snark and Watkins snark designs of orders 76 and 151.*

**Proof.** Let the vertex set of  $K_{76}$  be  $Z_{76}$ . The decompositions consist of

$(54, 47, 25, 16, 39, 41, 38, 22, 48, 32, 57, 58, 28, 14, 69, 59, 31, 27, 45, 46, 36, 37, 65, 26, 20, 44, 42, 4, 19, 23, 70, 12, 1, 67, 18, 10, 62, 11, 0, 29, 40, 35, 7, 60, 21, 64, 6, 66, 51, 74)_{\text{Sze}},$   
 $(22, 4, 66, 47, 59, 41, 44, 50, 53, 70, 8, 56, 65, 74, 3, 58, 61, 67, 28, 1, 20, 12, 38, 23, 0, 17, 2, 26, 43, 51, 9, 49, 63, 64, 16, 5, 29, 42, 25, 57, 35, 33, 11, 69, 73, 27, 71, 36, 68, 75)_{\text{Sze}}$   
and

$(35, 55, 62, 65, 49, 22, 17, 33, 21, 24, 26, 72, 60, 40, 6, 71, 38, 47, 48, 36, 45, 4, 7, 8, 67, 1, 23, 15, 34, 16, 64, 10, 29, 11, 59, 0, 18, 27, 66, 9, 70, 32, 39, 68, 42, 14, 46, 69, 3, 44)_{\text{Wat}},$   
 $(48, 22, 57, 28, 15, 19, 71, 10, 4, 40, 43, 74, 27, 63, 41, 49, 64, 30, 18, 68, 1, 52, 29, 26, 24, 9, 61, 32, 5, 37, 11, 51, 56, 17,$



39, 13, 53, 21, 6, 35, 2, 3, 42, 47, 54, 69, 70, 72, 34, 45)<sub>Wat</sub>  
under the action of the mapping  $x \mapsto x + 4 \pmod{76}$ .

Let the vertex set of  $K_{151}$  be  $Z_{151}$ . The decompositions consist of

(47, 54, 128, 27, 56, 15, 131, 51, 71, 99, 137, 21, 5, 23, 3, 29, 82,  
92, 24, 93, 129, 18, 132, 102, 104, 123, 33, 19, 110, 80, 105, 103, 100, 43,  
25, 121, 63, 97, 88, 72, 7, 49, 10, 14, 118, 57, 84, 89, 120, 2)<sub>Sze</sub>

and

(93, 6, 13, 146, 107, 110, 42, 144, 149, 46, 26, 61, 79, 140, 94, 100, 20,  
3, 75, 67, 97, 1, 134, 33, 27, 11, 72, 112, 21, 130, 120, 116, 53, 49,  
127, 92, 23, 129, 71, 121, 8, 50, 85, 128, 138, 39, 63, 118, 56, 5)<sub>Wat</sub>

under the action of the mapping  $x \mapsto x + 1 \pmod{151}$ .  $\square$

**Lemma 13.2** *There exist decompositions of  $K_{75,75,75}$  and  $K_{25,25,25,25}$  into each of the Szekeres snark and the Watkins snark.*

**Proof.** Let the vertex set of  $K_{75,75,75}$  be  $Z_{225}$  partitioned according to residue classes modulo 3. The decompositions consist of

(186, 132, 25, 92, 162, 42, 89, 205, 12, 214, 45, 28, 161, 100, 119, 34, 96,  
216, 175, 26, 152, 142, 157, 18, 36, 8, 101, 29, 30, 88, 217, 209, 192, 144,  
143, 53, 61, 22, 75, 134, 165, 49, 176, 82, 81, 39, 47, 196, 9, 173)<sub>Sze</sub>

and

(204, 53, 99, 109, 0, 161, 186, 139, 48, 4, 121, 185, 197, 70, 16, 141, 59,  
201, 49, 32, 30, 152, 223, 66, 165, 1, 205, 182, 7, 203, 89, 45, 87, 94,  
169, 208, 217, 143, 132, 209, 102, 17, 93, 23, 148, 145, 12, 177, 193, 158)<sub>Wat</sub>

under the action of the mapping  $x \mapsto x + 1 \pmod{225}$ .

Let the vertex set of  $K_{25,25,25,25}$  be  $Z_{100}$  partitioned according to residue classes modulo 4. The decompositions consist of

(32, 79, 37, 88, 71, 75, 81, 72, 53, 0, 14, 67, 50, 34, 48, 68, 45,  
87, 66, 57, 85, 43, 4, 77, 15, 78, 10, 30, 20, 23, 8, 25, 1, 7,  
56, 63, 12, 54, 94, 65, 80, 52, 35, 13, 11, 96, 62, 21, 83, 27)<sub>Sze</sub>,  
(12, 6, 87, 11, 40, 56, 72, 1, 98, 37, 79, 61, 86, 33, 18, 90, 96,  
92, 55, 14, 81, 67, 42, 52, 43, 73, 85, 54, 27, 49, 26, 7, 95, 97,  
38, 0, 62, 74, 28, 35, 23, 17, 80, 21, 65, 58, 10, 4, 9, 66)<sub>Sze</sub>

and

(10, 53, 36, 1, 3, 59, 46, 67, 96, 17, 14, 75, 23, 45, 25, 64, 43,  
89, 74, 32, 84, 66, 0, 6, 51, 85, 48, 9, 91, 18, 80, 27, 44, 82,  
37, 49, 79, 30, 86, 83, 63, 12, 54, 11, 87, 13, 93, 26, 92, 71)<sub>Wat</sub>,  
(75, 26, 5, 40, 92, 67, 31, 9, 14, 13, 81, 34, 47, 1, 70, 0, 23,  
82, 58, 42, 93, 87, 74, 44, 91, 72, 4, 18, 29, 21, 66, 84, 88, 85,  
41, 12, 7, 15, 8, 90, 60, 6, 20, 53, 24, 37, 50, 97, 35, 28)<sub>Wat</sub>

under the action of the mapping  $x \mapsto x + 4 \pmod{100}$ .  $\square$

Theorem 1.6 (vii) follows from Lemmas 13.1, 13.2 and Proposition 1.4. The proof of Theorem 1.6 is complete.

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